ON THE HOMOLOGY OF COMPLETION AND TORSION

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Abstract. Let A be a commutative ring, and $\mathfrak{a}$ a *weakly proregular* ideal in A. This includes the noetherian case: if A is noetherian then any ideal in it is weakly proregular; but there are other interesting examples. In this paper we prove the *MGM equivalence*, which is an equivalence between the category of *cohomologically $\mathfrak{a}$-adically complete complexes* and the category of *cohomologically $\mathfrak{a}$-torsion complexes*. These are triangulated subcategories of the derived category of A-modules. Our work extends earlier work by Alonso-Jeremias-Lipman, Schenzel and Dwyer-Greenlees.

Contents

0. Introduction

Let A be a commutative ring, and let $\mathfrak{a}$ be an ideal in it. (We do not assume that A is noetherian or $\mathfrak{a}$-adically complete.) There are two operations associated to this data: the *$\mathfrak{a}$-adic completion* and the *$\mathfrak{a}$-torsion*. For an A-module M its $\mathfrak{a}$-adic completion is the A-module

$$\Lambda_{\mathfrak{a}}(M) = \widehat{M} := \lim_{\leftarrow i} M/\mathfrak{a}^i M.$$

An element $m \in M$ is called an $\mathfrak{a}$-torsion element if $\mathfrak{a}^i m = 0$ for $i \gg 0$. The $\mathfrak{a}$-torsion elements form the $\mathfrak{a}$-torsion submodule $\Gamma_{\mathfrak{a}}(M)$ of M.

Let us denote by $\mathsf{Mod}\, A$ the category of A-modules. So we have additive functors

$$\Lambda_{\mathfrak{a}}, \Gamma_{\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A.$$

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The functor $\Gamma_{\mathfrak{a}}$ is left exact; whereas $\Lambda_{\mathfrak{a}}$ is neither left exact nor right exact. (Of course when A is noetherian, the completion functor $\Lambda_{\mathfrak{a}}$ is exact on the subcategory $\mathsf{Mod}_{\mathrm{f}}\, A$ of finitely generated modules.) In this paper we study several questions of homological nature about these two functors.

The derived category of $\mathsf{Mod}\, A$ is denoted by $\mathsf{D}(\mathsf{Mod}\, A)$. As explained in Section 1, the derived functors

$$\mathrm{L}\Lambda_{\mathfrak{a}}, \mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

exist. The left derived functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ is constructed using K-projective resolutions, and the right derived functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ is constructed using K-injective resolutions.

The functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ has been studied in great length already in the 1950's, by Grothendieck and others (in the context of local cohomology).

The left derived functors $\mathrm{L}^i\Lambda_{\mathfrak{a}}$ were studied by Matlis [Ma2] and Greenlees-May [GM]. The first treatment of the total left derived functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ was in the paper [AJL1] by Alonso-Jeremias-Lipman from 1997. In this paper the authors established the *Greenlees-May Duality*, which we find deep and remarkable. The setting in [AJL1] is geometric: the completion of a non-noetherian scheme along a proregularly embedded closed subset. However, certain aspects of the theory remained unclear (see Remarks 4.28 and 6.14). One of our aims in this paper is to clarify the foundations of the theory in the algebraic setting. We also extend the scope of the existing results.

Two other, much more recent papers also influenced our work. In the paper [KS2] of Kashiwara-Schapira there is a part devoted to what they call *cohomologically complete complexes*. We wondered what might be the relation between this notion and the derived completion functor $\mathrm{L}\Lambda_{\mathfrak{a}}$. The answer we discovered is Theorem 0.6 below.

The paper [Ef] by Efimov describes an operation of *completion by derived double centralizer*. This idea is attributed to Kontsevich. A similar results was obtained in [DGI]. Our interpretation of this completion operation is in the companion paper [PSY1], and it relies on the work in this paper.

Let us turn to the results in our paper. We work in the following context: A is a commutative ring, and $\mathfrak{a}$ is a *weakly proregular ideal* in it. By definition an ideal is weakly proregular if it can be generated by a *weakly proregular sequence* $\boldsymbol{a} = (a_1, \ldots, a_n)$ of elements of A. The definition of proregularity for sequences is a bit technical (see Definition 3.21). It is important to know that:

Theorem 0.1 ([Sc])**.** *If A is a noetherian commutative ring, then every finite sequence in A is weakly proregular, and every ideal in A is weakly proregular.*

We provide a short proof of this for the benefit of the reader (see Theorem 3.34 in the body of the paper). We also give a fairly natural example of a weakly proregular sequence in a non-noetherian ring (Example 3.35). A useful fact is in the following theorem (which is Corollary 5.2 in the body of the paper).

Theorem 0.2. *Let $\mathfrak{a}$ be a weakly proregular ideal in a ring A. Then any finite sequence that generates $\mathfrak{a}$ is weakly proregular.*

A complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ is called a *cohomologically $\mathfrak{a}$-torsion complex* if the canonical morphism $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism. The complex M is called a *cohomologically $\mathfrak{a}$-adically complete complex* if the canonical morphism $M \to \mathrm{L}\Lambda_{\mathfrak{a}}(M)$ is an isomorphism. We denote by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$ and $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$

the full subcategories of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of cohomologically $\mathfrak{a}$-torsion complexes and cohomologically $\mathfrak{a}$-adically complete complexes, respectively. These are triangulated subcategories.

Here is the main result of our paper.

Theorem 0.3 (MGM Equivalence)**.** *Let A be a commutative ring, and $\mathfrak{a}$ a weakly proregular ideal in it.*

(1) *For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ one has $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}(M) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$.*

(2) *The functor*

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}} \to \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$$

is an equivalence, with quasi-inverse $\mathrm{L}\Lambda_{\mathfrak{a}}$.

This is repeated as Theorem 6.11 in the body of the paper. The letters “MGM” stand for Matlis, Greenlees and May.

Similar results can be found in [AJL1, Sc, DG], and possibly some weaker version of Theorem 0.3 can be deduced from these results. But as far as we can tell, Theorem 0.3 is new. See Remarks 4.28 and 6.14 for a discussion. The main ingredient in the proof of the MGM equivalence is Theorem 0.4 below.

Given a finite sequence $\boldsymbol{a}$ that generates $\mathfrak{a}$, we construct explicitly a complex $\mathrm{Tel}(A; \boldsymbol{a})$, called the *telescope complex*. It is a bounded complex of countable rank free A-modules. There is a functorial homomorphism of complexes (also with explicit formula)

$$\mathrm{tel}_{\boldsymbol{a},M} : \mathrm{Hom}_A\big(\mathrm{Tel}(A; \boldsymbol{a}), M\big) \to \Lambda_{\mathfrak{a}}(M)$$

for any $M \in \mathsf{Mod}\, A$. By totalization we get a homomorphism $\mathrm{tel}_{\boldsymbol{a},M}$ for any $M \in \mathsf{C}(\mathsf{Mod}\, A)$. See Definitions 4.1 and 4.16.

Theorem 0.4. *Let A be a commutative ring, let $\boldsymbol{a}$ be a weakly proregular sequence in A, and let $\mathfrak{a}$ be the ideal generated by $\boldsymbol{a}$. If P is a K-flat complex of A-modules, then the homomorphism*

$$\mathrm{tel}_{\boldsymbol{a},P} : \mathrm{Hom}_A\big(\mathrm{Tel}(A; \boldsymbol{a}), P\big) \to \Lambda_{\mathfrak{a}}(P)$$

is a quasi-isomorphism.

This is Corollary 4.23 in the body of the paper. The concept of telescope complex is not new of course, but our treatment appears to be quite different from anything we saw in the literature.

Along the way we also prove that the functors $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}$ have finite cohomological dimensions. (An upper bound is the minimal length of a sequence that generates the ideal $\mathfrak{a}$.) This implies that

$$\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}} = \mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A), \tag{0.5}$$

the latter being the subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of complexes with $\mathfrak{a}$-torsion cohomology modules (see Corollary 3.32). Note that such a statement for $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$ is false: in Example 3.33 we exhibit a cohomologically $\mathfrak{a}$-adically complete complex P such that $\mathrm{H}^i(P) = 0$ for all $i \neq 0$, and the module $\mathrm{H}^0(P)$ is *not* $\mathfrak{a}$-adically complete.

Let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a generating sequence for the ideal $\mathfrak{a}$. In Section 7 we construct a noncommutative DG A-algebra $\mathrm{C}(A; \boldsymbol{a})$, which we call the *derived localization of A with respect to $\boldsymbol{a}$*. When $n = 1$ (we refer to this as the principal

case, since the ideal $\mathfrak{a}$ is principal) then $\mathrm{C}(A;\boldsymbol{a}) = A[a_1^{-1}]$, the usual localization. For $n > 1$ the construction uses the Čech cosimplicial algebra and the Alexander-Whitney multiplication.

Theorem 0.6. *Let A be a commutative ring, $\boldsymbol{a}$ a weakly proregular sequence in A, and $\mathfrak{a}$ the ideal generated by $\boldsymbol{a}$. The following conditions are equivalent for $M \in \mathsf{D}(\mathsf{Mod}\, A)$:*

(i) *M is cohomologically $\mathfrak{a}$-adically complete.*
(ii) $\mathrm{RHom}_A\big(\mathrm{C}(A;\boldsymbol{a}), M\big) = 0$.

This is Theorem 7.6 in the body of the paper. The principal regular case ($n = 1$ and a_1 a non-zero-divisor) was considered by Kashiwara-Schapira in [KS2]. Indeed, in [KS2] condition (ii) was used as the definition of cohomologically complete complexes. After hearing about the results of [KS2], we wondered whether they hold in greater generality (for $n > 1$ and no regularity assumption on the sequence $\boldsymbol{a}$). More in this direction can be found in the companion paper [PSY2].

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1. Preliminaries on Homological Algebra

This paper relies on delicate work with derived functors. Therefore we begin with a review of some facts on homological algebra. There are also a few new results. By default all rings considered in the paper are commutative.

Let M be an abelian category. As in [RD] we denote by $\mathsf{C}(\mathsf{M})$ the category of complexes of objects of M, by $\mathsf{K}(\mathsf{M})$ its homotopy category, and by $\mathsf{D}(\mathsf{M})$ the derived category. There are full subcategories $\mathsf{D}^-(\mathsf{M})$, $\mathsf{D}^+(\mathsf{M})$ and $\mathsf{D}^{\mathrm{b}}(\mathsf{M})$ of $\mathsf{D}(\mathsf{M})$, whose objects are the bounded above, bounded below and bounded complexes respectively.

Our notation for distinguished triangles in $\mathsf{K}(\mathsf{M})$ or $\mathsf{D}(\mathsf{M})$ is either $L \xrightarrow{\alpha} M \xrightarrow{\beta} N \xrightarrow{\gamma} L[1]$, or simply $L \to M \to N \xrightarrow{\triangle}$ if the names of the morphisms are not important.

A complex $P \in \mathsf{C}(\mathsf{M})$ is called *K-projective* if for any acyclic complex $N \in \mathsf{C}(\mathsf{M})$ the complex $\mathrm{Hom}_{\mathsf{M}}(P, N)$ is also acyclic. A complex $I \in \mathsf{C}(\mathsf{M})$ is called *K-injective* if for any acyclic complex $N \in \mathsf{C}(\mathsf{M})$ the complex $\mathrm{Hom}_{\mathsf{M}}(N, I)$ is also acyclic. These definitions were introduced in [Sp]; in [Ke, Section 3] it is shown that "K-projective" is the same as "having property (P)", and "K-injective" is the same as "having property (I)".

A K-projective resolution of $M \in \mathsf{C}(\mathsf{M})$ is a quasi-isomorphism $P \to M$ in $\mathsf{C}(\mathsf{M})$ with P a K-projective complex. If every $M \in \mathsf{C}(\mathsf{M})$ admits some K-projective resolution, then we say that $\mathsf{C}(\mathsf{M})$ has enough K-projectives. Similarly for K-injectives.

Now we specialize to the case $\mathsf{M} := \mathsf{Mod}\, A$, where A is a ring. A complex $P \in \mathsf{C}(\mathsf{Mod}\, A)$ is called *K-flat* if for any acyclic complex $N \in \mathsf{C}(\mathsf{Mod}\, A)$ the complex $N \otimes_A P$ is also acyclic. Note that a K-projective complex P is K-flat.

Here is a useful existence result.

Proposition 1.1. *Let A be a ring, and let $M \in \mathsf{C}(\mathsf{Mod}\, A)$.*

(1) *The complex M admits a quasi-isomorphism $P \to M$, where P is a K-projective complex, and moreover each component P^i is a free A-module.*

(2) *The complex M admits a quasi-isomorphism $P \to M$, where P is a K-flat complex, and moreover each component P^i is a flat A-module.*
(3) *The complex M admits a quasi-isomorphism $M \to I$, where I is a K-injective complex, and moreover each component I^i is an injective A-module.*

Proof. (1) This is proved in [Ke, Subsection 3.1], when discussing the existence of P-resolutions. Cf. [Sp, Corollary 3.5].

(2) This follows from (1), since any K-projective complex is also K-flat.

(3) See [Ke, Subsection 3.2]. Cf. [Sp, Proposition 3.11]. □

In particular, the proposition says that $\mathsf{C}(\mathsf{Mod}\, A)$ has enough K-projectives, K-flats and K-injectives.

Remark 1.2. Let $(X, \mathcal{A})$ be a ringed space, and let $\mathsf{Mod}\,\mathcal{A}$ be the category of sheaves of $\mathcal{A}$-modules. It is known that $\mathsf{C}(\mathsf{Mod}\,\mathcal{A})$ has enough K-injectives and enough K-flats; but their structure is more complicated than in the case of $\mathsf{C}(\mathsf{Mod}\, A)$, and Proposition 1.1 might not hold.

Here are a few facts about K-projective and K-injective resolutions, compiled from [Sp, BN, Ke]. The first are: a bounded above complex of projectives is K-projective, a bounded above complex of flats is K-flat, and a bounded below complex of injectives is K-injective.

Once again M is an abelian category. Let E be some triangulated category, and let $F : \mathsf{K}(\mathsf{M}) \to \mathsf{E}$ be a triangulated functor. If $\mathsf{C}(\mathsf{M})$ has enough K-projectives, then the left derived functor $(\mathrm{L}F, \xi) : \mathsf{D}(\mathsf{M}) \to \mathsf{E}$ exists, and it is calculated by K-projective resolutions. Likewise, if $\mathsf{K}(\mathsf{M})$ has enough K-injectives, then the right derived functor $(\mathrm{R}F, \xi) : \mathsf{D}(\mathsf{M}) \to \mathsf{E}$ exists, and it is calculated by K-injective resolutions.

Let $M = \{M^i\}_{i \in \mathbb{Z}}$ be a graded object of M. We define

$$\inf(M) := \inf\{i \mid M^i \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\} \tag{1.3}$$

and

$$\sup(M) := \sup\{i \mid M^i \neq 0\} \in \mathbb{Z} \cup \{\pm\infty\}. \tag{1.4}$$

The amplitude of M is

$$\operatorname{amp}(M) := \sup(M) - \inf(M) \in \mathbb{N} \cup \{\pm\infty\}. \tag{1.5}$$

(For $M = 0$ this reads $\inf(M) = \infty$, $\sup(M) = -\infty$ and $\operatorname{amp}(M) = -\infty$.) Thus M is bounded iff $\operatorname{amp}(M) < \infty$.

For $M \in \mathsf{D}(\mathsf{M})$ we write $\mathrm{H}(M) := \{\mathrm{H}^i(M)\}_{i \in \mathbb{Z}}$.

Definition 1.6. Let M and M' be abelian categories, and let $F : \mathsf{D}(\mathsf{M}) \to \mathsf{D}(\mathsf{M}')$ be a triangulated functor. Let $\mathsf{E} \subset \mathsf{D}(\mathsf{M})$ be a full additive subcategory (not necessarily triangulated), and consider the restricted functor $F|_{\mathsf{E}} : \mathsf{E} \to \mathsf{D}(\mathsf{M}')$.

(1) We say that $F|_{\mathsf{E}}$ has *finite cohomological dimension* if there exist some $n \in \mathbb{N}$ and $s \in \mathbb{Z}$ such that for every complex $M \in \mathsf{E}$ one has

$$\sup\big(\mathrm{H}(F(M))\big) \leq \sup\big(\mathrm{H}(M)\big) + s$$

and

$$\inf\big(\mathrm{H}(F(M))\big) \geq \inf\big(\mathrm{H}(M)\big) + s - n.$$

The smallest such number n is called the *cohomological dimension* of $F|_{\mathsf{E}}$.

(2) If no such n and s exist then we say $F|_{\mathsf{E}}$ has *infinite cohomological dimension.*

The number s appearing in the definition represents the shift. (An easy calculation shows that if $F|_{\mathsf{E}}$ is nonzero and has finite cohomological dimension n, then the shift s in the definition is unique.)

If the functor F has finite cohomological dimension, then it is a *way-out functor in both directions*, in the sense of [RD, Section I.7]. We will use this fact several times.

Example 1.7. Take a nonzero ring A, and let $P := A[1] \oplus A[2]$, a complex with zero differential concentrated in degrees -1 and -2. The functor $F := P \otimes_A -$ has cohomological dimension $n = 1$, with shift $s = -1$.

Proposition 1.8. *Let M, M' and M'' be abelian categories, and let $F : \mathsf{D}(\mathsf{M}) \to \mathsf{D}(\mathsf{M}')$ and $F' : \mathsf{D}(\mathsf{M}') \to \mathsf{D}(\mathsf{M}'')$ be triangulated functors. Assume the cohomological dimensions of F and F' are n and n' respectively. Then the cohomological dimension of $F' \circ F$ is at most $n + n'$.*

We leave out the easy proof.

Here is a useful criterion for quasi-isomorphisms (a variant of the way-out argument). For $i, j \in \mathbb{Z}$ let $\mathsf{C}^{[i,j]}(\mathsf{M})$ be the full subcategory of $\mathsf{C}(\mathsf{M})$ whose objects are the complexes concentrated in the degree range $[i, j] := \{i, i+1, \ldots, j\}$.

Proposition 1.9. *Let M and M' be abelian categories, let $F, G : \mathsf{M} \to \mathsf{C}(\mathsf{M}')$ be additive functors, and let $\eta : F \to G$ be a natural transformation. Assume M' has countable direct sums, and consider the extensions $F, G : \mathsf{C}(\mathsf{M}) \to \mathsf{C}(\mathsf{M}')$ by the direct sum totalization. Suppose $M \in \mathsf{C}(\mathsf{M})$ satisfies these two conditions*:

(i) *There are $j_0, j_1 \in \mathbb{Z}$ such that $F(M^i), G(M^i) \in \mathsf{C}^{[j_0,j_1]}(\mathsf{M}')$ for every $i \in \mathbb{Z}$.*
(ii) *The homomorphism $\eta_{M^i} : F(M^i) \to G(M^i)$ is a quasi-isomorphism for every $i \in \mathbb{Z}$.*

Then $\eta_M : F(M) \to G(M)$ is a quasi-isomorphism.

Proof. Step 1. Assume that M is bounded. We prove that η_M is a quasi-isomorphism by induction on $\operatorname{amp}(M)$. If $\operatorname{amp}(M) = 0$ then this is given. The inductive step is done using the stupid truncation functors

$$\operatorname{stt}^{>i}(M), \operatorname{stt}^{\leq i}(M) : \mathsf{C}(\mathsf{M}) \to \mathsf{C}(\mathsf{M}), \tag{1.10}$$

and the related short exact sequences. See [RD, pages 69-70], where the truncations $\operatorname{stt}^{>i}(M)$ and $\operatorname{stt}^{\leq i}(M)$ are denoted by $\tau_{>i}(M)$ and $\tau_{\leq i}(M)$ respectively.

Step 2. Now M is arbitrary. We have to prove that $\mathrm{H}^i(\eta_M) : \mathrm{H}^i(F(M)) \to \mathrm{H}^i(G(M))$ is an isomorphism for every $i \in \mathbb{Z}$. For any $i \leq j$ there is the double truncation functor $\operatorname{stt}^{[i,j]} := \operatorname{stt}^{\leq j} \circ \operatorname{stt}^{>i}$. So let us fix i. The homomorphism $\mathrm{H}^i(\eta_M)$ in M' only depends on the homomorphism of complexes

$$\operatorname{stt}^{[i-1,i+1]}(\eta_M) : \operatorname{stt}^{[i-1,i+1]}(F(M)) \to \operatorname{stt}^{[i-1,i+1]}(G(M)).$$

Therefore we can replace η_M with $\eta_{M'} : F(M') \to G(M')$, where

$$M' := \operatorname{stt}^{[j_0+i-1, j_1+i+1]}(M).$$

But M' is bounded, so by part (1) the homomorphism $\eta_{M'}$ is a quasi-isomorphism. ∎

To end this section, here is a basic result we need, that we could not locate in the literature (but that was used implicitly in [Sc]).

Proposition 1.11. *Let* M *and* N *be abelian categories, let* $F : \mathsf{M} \to \mathsf{N}$ *be an exact additive covariant functor, and let* $G : \mathsf{M} \to \mathsf{N}$ *be an exact additive contravariant functor. Then for any* $M \in \mathsf{C}(\mathsf{M})$ *there are isomorphisms* $\mathrm{H}^k(F(M)) \cong F(\mathrm{H}^k(M))$ *and* $\mathrm{H}^{-k}(G(M)) \cong G(\mathrm{H}^k(M))$ *in* N*. Moreover, these isomorphisms are functorial in* M*,* F *and* G*.*

Proof. These are very degenerate cases of Grothendieck spectral sequences. Here is a direct proof. We use the notation $\mathrm{Z}^k(M) := \mathrm{Ker}(\mathrm{d} : M^k \to M^{k+1})$, which is the object of k-cocycles of M, and $\mathrm{Y}^k(M) := \mathrm{Coker}(\mathrm{d} : M^{k-1} \to M^k)$, which does not have a name. There are functorial isomorphisms

$$\mathrm{H}^k(M) \cong \mathrm{Coker}\big(\mathrm{d} : M^{k-1} \to \mathrm{Z}^k(M)\big) \cong \mathrm{Ker}\big(\mathrm{d} : \mathrm{Y}^k(M) \to M^{k+1}\big).$$

For any additive functor $F : \mathsf{M} \to \mathsf{N}$ (not necessarily exact) there is an obvious morphism $\alpha : F(\mathrm{Z}^k(M)) \to \mathrm{Z}^k(F(M))$, and it induces a morphism $\bar{\alpha} : F(\mathrm{H}^k(M)) \to \mathrm{H}^k(F(M))$. An easy calculation shows that when F is exact, the morphisms α and $\bar{\alpha}$ are isomorphisms.

Given a contravariant additive functor $G : \mathsf{M} \to \mathsf{N}$, there is a morphism (slightly less obvious than α, because of the change in direction) $\beta : G(\mathrm{Y}^k(M)) \to \mathrm{Z}^{-k}(G(M))$. This induces a morphism $\bar{\beta} : G(\mathrm{H}^k(M)) \to \mathrm{H}^{-k}(G(M))$. When G is exact, the morphisms β and $\bar{\beta}$ are isomorphisms. □

Corollary 1.12. *Let* A *be a ring,* M *a complex of* A*-modules,* P *a flat* A*-module, and* I *and injective* A*-module. There are isomorphisms*

$$\mathrm{H}^k(M \otimes_A P) \cong \mathrm{H}^k(M) \otimes_A P$$

and

$$\mathrm{H}^{-k}(\mathrm{Hom}_A(M, I)) \cong \mathrm{Hom}_A(\mathrm{H}^k(M), I),$$

functorial in M, P *and* I*.*

Proof. Take $F(M) := M \otimes_A P$ and $G(M) := \mathrm{Hom}_A(M, I)$, and use the proposition above. (These are degenerate cases of the Künneth Theorems.) □

2. The Derived Completion and Torsion Functors

In this section A is a commutative ring, and $\mathfrak{a}$ is an ideal in it. We do not assume that $\mathfrak{a}$ is finitely generated or that A is $\mathfrak{a}$-adically complete.

For any $i \in \mathbb{N}$ let $A_i := A/\mathfrak{a}^{i+1}$. The collection of rings $\{A_i\}_{i \in \mathbb{N}}$ forms an inverse system. Following [GM, AJL1], for an A-module M we write

$$\Lambda_{\mathfrak{a}}(M) := \lim_{\leftarrow i} (A_i \otimes_A M) \tag{2.1}$$

for the $\mathfrak{a}$-adic completion of M, although we sometimes use the more conventional (yet possibly ambiguous) notation $\widehat{M}$. We get an additive functor $\Lambda_{\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$. Recall that there is a functorial homomorphism

$$\tau_M : M \to \Lambda_{\mathfrak{a}}(M) \tag{2.2}$$

for $M \in \mathsf{Mod}\, A$, coming from the homomorphisms $M \to A_i \otimes_A M$. The module M is called $\mathfrak{a}$-*adically complete* if τ_M is an isomorphism. (Some texts, such as [Bo], would say that M is separated and complete). As customary, when M is complete we usually identify M with $\Lambda_{\mathfrak{a}}(M)$ via τ_M.

If the ideal $\mathfrak{a}$ is finitely generated, then the functor $\Lambda_{\mathfrak{a}}$ is idempotent, in the sense that the homomorphism

$$\tau_{\Lambda_{\mathfrak{a}}(M)} : \Lambda_{\mathfrak{a}}(M) \to \Lambda_{\mathfrak{a}}(\Lambda_{\mathfrak{a}}(M))$$

is an isomorphism for every module M (see [Ye2, Corollary 3.6]).

Let $\widehat{A} := \Lambda_{\mathfrak{a}}(A)$. Then $\widehat{A}$ is a ring, and $\tau_A : A \to \widehat{A}$ is a ring homomorphism. If A is noetherian then $\widehat{A}$ is also noetherian, and flat over A. One can view the completion as a functor $\Lambda_{\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, \widehat{A}$. But in this paper we shall usually ignore this.

Remark 2.3. The full subcategory of $\mathsf{Mod}\, A$ consisting of $\mathfrak{a}$-adically complete modules is additive, but not abelian in general.

It is well known that when A is noetherian, the completion functor $\Lambda_{\mathfrak{a}}$ is exact on $\mathsf{Mod}_{\mathrm{f}}\, A$, the category of finitely generated modules. However, on $\mathsf{Mod}\, A$ the functor $\Lambda_{\mathfrak{a}}$ is neither left exact nor right exact, even in the noetherian case (see [Ye2, Examples 3.19 and 3.20]).

When A is not noetherian, we do not know if $\widehat{A}$ is flat over A. Still, if $\mathfrak{a}$ is finitely generated, and we let $\widehat{\mathfrak{a}} := \widehat{A}\mathfrak{a} \subset \widehat{A}$, then $\widehat{A}$ is $\widehat{\mathfrak{a}}$-adically complete; this follows from [Ye2, Corollary 3.6].

If the ideal $\mathfrak{a}$ is not finitely generated, things are even worse: the functor $\Lambda_{\mathfrak{a}}$ can fail to be idempotent; i.e. the completion $\Lambda_{\mathfrak{a}}(M)$ of a module M could fail to be complete. See [Ye2, Example 1.8].

As for any additive functor, the functor $\Lambda_{\mathfrak{a}}$ has a left derived functor

$$\mathrm{L}\Lambda_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A) \ , \ \xi : \mathrm{L}\Lambda_{\mathfrak{a}} \to \Lambda_{\mathfrak{a}} \tag{2.4}$$

constructed using K-projective resolutions.

The next result was proved in [AJL1]. Since this is so fundamental, we chose to reproduce the easy proof.

Lemma 2.5 ([AJL1])**.** *Let P be an acyclic K-flat complex of A-modules. Then the complex $\Lambda_{\mathfrak{a}}(P)$ is also acyclic.*

Proof. Since P is both acyclic and K-flat, for any i we have an acyclic complex $A_i \otimes_A P$. The collection of complexes $\{A_i \otimes_A P\}_{i\in\mathbb{N}}$ is an inverse system, and the homomorphism $A_{i+1} \otimes_A P^j \to A_i \otimes_A P^j$ is surjective for every i and j. But $\Lambda_{\mathfrak{a}}(P^j) = \lim_{\leftarrow i} (A_i \otimes_A P^j)$. By the Mittag-Leffler argument (see [KS1, Proposition 1.12.4] or [We, Theorem 3.5.8]) the complex $\Lambda_{\mathfrak{a}}(P)$ is acyclic. □

Proposition 2.6. *If P is a K-flat complex then the morphism $\xi_P : \mathrm{L}\Lambda_{\mathfrak{a}}(P) \to \Lambda_{\mathfrak{a}}(P)$ in $\mathsf{D}(\mathsf{Mod}\, A)$ is an isomorphism. Thus we can calculate $\mathrm{L}\Lambda_{\mathfrak{a}}$ using K-flat resolutions.*

Proof. This is immediate from Lemma 2.5; cf. [RD, Theorem I.5.1]. □

Proposition 2.7 ([AJL1])**.** *Let $M \in \mathsf{D}(\mathsf{Mod}\, A)$. There is a morphism $\tau^{\mathrm{L}}_M : M \to \mathrm{L}\Lambda_{\mathfrak{a}}(M)$ in $\mathsf{D}(\mathsf{Mod}\, A)$, functorial in M, such that $\xi_M \circ \tau^{\mathrm{L}}_M = \tau_M$ as morphisms $M \to \Lambda_{\mathfrak{a}}(M)$.*

Proof. Given $M \in \mathsf{D}(\mathsf{Mod}\, A)$ let us choose a K-projective resolution $\phi : P \to M$. Since ϕ and ξ_P are isomorphisms in $\mathsf{D}(\mathsf{Mod}\, A)$, we can define

$$\tau^{\mathrm{L}}_M := \mathrm{L}\Lambda_{\mathfrak{a}}(\phi) \circ \xi_P^{-1} \circ \tau_P \circ \phi^{-1} : M \to \mathrm{L}\Lambda_{\mathfrak{a}}(M).$$

This is independent of the the chosen resolution ϕ, and satisfies $\xi_M \circ \tau_M = \tau^{\mathrm{L}}_M$. □

Definition 2.8. (1) A complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ is called $\mathfrak{a}$-*adically cohomologically complete* if the morphism $\tau^{\mathrm{L}}_M : M \to \mathrm{L}\Lambda_{\mathfrak{a}}(M)$ is an isomorphism.
(2) The full subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of $\mathfrak{a}$-adically cohomologically complete complexes is denoted by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$.

It is clear that the subcategory $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$ is triangulated.

The notion of cohomologically complete complex is quite illusive. See Example 3.33 .

For an A-module M and $i \in \mathbb{N}$ we identify $\mathrm{Hom}_A(A_i, M)$ with the submodule

$$\{m \in M \mid \mathfrak{a}^{i+1} m = 0\} \subset M.$$

The $\mathfrak{a}$-*torsion submodule* of M is

$$\Gamma_{\mathfrak{a}}(M) := \bigcup_{i \in \mathbb{N}} \mathrm{Hom}_A(A_i, M) \subset M.$$

The module M is called an $\mathfrak{a}$-*torsion module* if $\Gamma_{\mathfrak{a}}(M) = M$. We denote by $\mathsf{Mod}_{\mathfrak{a}\text{-tor}}\, A$ the full subcategory of $\mathsf{Mod}\, A$ consisting of $\mathfrak{a}$-torsion modules.

We get an additive functor $\Gamma_{\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A$. In fact this is a left exact functor. There is a functorial homomorphism $\sigma_M : \Gamma_{\mathfrak{a}}(M) \to M$ which is just the inclusion. The functor $\Gamma_{\mathfrak{a}}$ is idempotent, in the sense that $\sigma_{\Gamma_{\mathfrak{a}}(M)} : \Gamma_{\mathfrak{a}}(\Gamma_{\mathfrak{a}}(M)) \to \Gamma_{\mathfrak{a}}(M)$ is bijective.

Like every additive functor, the functor $\Gamma_{\mathfrak{a}}$ has a right derived functor

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)\ ,\ \xi : \Gamma_{\mathfrak{a}} \to \mathrm{R}\Gamma_{\mathfrak{a}} \tag{2.9}$$

constructed using K-injective resolutions.

Proposition 2.10. *There is a functorial morphism $\sigma^{\mathrm{R}}_M : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$, such that $\sigma_M = \sigma^{\mathrm{R}}_M \circ \xi_M$ as morphisms $\Gamma_{\mathfrak{a}}(M) \to M$ in $\mathsf{D}(\mathsf{Mod}\, A)$.*

Proof. Choose a K-injective resolution $\phi : M \to I$, and define

$$\sigma^{\mathrm{R}}_M := \phi^{-1} \circ \sigma_I \circ \xi_I^{-1} \circ \mathrm{R}\Gamma_{\mathfrak{a}}(\phi).$$

This is independent of the resolution. □

Definition 2.11. (1) A complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ is called *cohomologically $\mathfrak{a}$-torsion* if the morphism $\sigma^{\mathrm{R}}_M : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism.
(2) The full subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of cohomologically $\mathfrak{a}$-torsion complexes is denoted by $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$.
(3) We denote by $\mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A)$ the full subcategory of $\mathsf{D}(\mathsf{Mod}\, A)$ consisting of the complexes whose cohomology modules are in $\mathsf{Mod}_{\mathfrak{a}\text{-tor}}\, A$.

It is clear that the subcategory $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$ is triangulated.

Since $\mathsf{Mod}_{\mathfrak{a}\text{-tor}}\, A$ is a thick abelian subcategory of $\mathsf{Mod}\, A$, it follows that $\mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A)$ is a triangulated category. Note that $\Gamma_{\mathfrak{a}}(I) \in \mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A)$ for any K-injective complex I. Therefore

$$\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tors}} \subset \mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A). \tag{2.12}$$

Later (in Corollary 3.32) we shall see that there is equality in (2.12) under some extra assumption.

3. Koszul Complexes and Weak Proregularity

In this section we define *weakly proregular sequences*. We also set up notation to be used later. The definitions and some of the results in this section are contained in [AJL1] and [Sc]. We have included our own short proofs, for the benefit of the reader. We also give a new motivating example at the end.

Let A be a commutative ring (not necessarily noetherian). Recall that for an element $a \in A$ the *Koszul complex* $\mathrm{K}(A;a)$ is the complex

$$\mathrm{K}(A;a) := \big(\cdots \to 0 \to A \xrightarrow{a\cdot} A \to 0 \to \cdots \big) \tag{3.1}$$

concentrated in degrees -1 and 0. Now let $\boldsymbol{a} = (a_1, \ldots, a_n)$ be a sequence of elements of A. The Koszul complex associated to $\boldsymbol{a}$ is the complex of A-modules

$$\mathrm{K}(A;\boldsymbol{a}) := \mathrm{K}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{K}(A;a_n). \tag{3.2}$$

Observe that $\mathrm{K}(A;\boldsymbol{a})^0 \cong A$, and $\mathrm{K}(A;\boldsymbol{a})^{-1}$ is a free A-module of rank n. Moreover, $\mathrm{K}(A;\boldsymbol{a})$ is a super-commutative DG algebra: as a graded algebra it is the exterior algebra over A of the module $\mathrm{K}(A;\boldsymbol{a})^{-1}$. There is a DG algebra homomorphism

$$e_{\boldsymbol{a}} : A \to \mathrm{K}(A;\boldsymbol{a}). \tag{3.3}$$

Let us denote by $(\boldsymbol{a})$ the ideal generated by the sequence $\boldsymbol{a}$, so that

$$A/(\boldsymbol{a}) \cong A/(a_1) \otimes_A \cdots \otimes_A A/(a_n)$$

as A-algebras. There is an A-algebra isomorphism

$$\mathrm{H}^0(\mathrm{K}(A;\boldsymbol{a})) \cong A/(\boldsymbol{a}). \tag{3.4}$$

For any $j \geq i$ in $\mathbb{N}$ there is a homomorphism of complexes

$$p_{a,j,i} : \mathrm{K}(A;a^j) \to \mathrm{K}(A;a^i), \tag{3.5}$$

which is the identity in degree 0, and multiplication by a^{j-i} in degree -1. This operation makes sense also for sequences: given a sequence $\boldsymbol{a}$ as above, let us write $\boldsymbol{a}^i := (a_1^i, \ldots, a_n^i)$. There is a homomorphism of complexes

$$p_{\boldsymbol{a},j,i} : \mathrm{K}(A;\boldsymbol{a}^j) \to \mathrm{K}(A;\boldsymbol{a}^i)\ ,\ p_{\boldsymbol{a},j,i} := p_{a_1,j,i} \otimes \cdots \otimes p_{a_n,j,i}. \tag{3.6}$$

In fact $p_{\boldsymbol{a},j,i}$ is a homomorphism of DG algebras, and $\mathrm{H}^0(p_{\boldsymbol{a},j,i})$ corresponds via (3.4) to the canonical surjection $A/(\boldsymbol{a}^j) \to A/(\boldsymbol{a}^i)$. The homomorphisms

$$\mathrm{H}^k(p_{\boldsymbol{a},j,i}) : \mathrm{H}^k(\mathrm{K}(A;\boldsymbol{a}^j)) \to \mathrm{H}^k(\mathrm{K}(A;\boldsymbol{a}^i)) \tag{3.7}$$

make $\big\{ \mathrm{H}^k(\mathrm{K}(A;\boldsymbol{a}^i)) \big\}_{i \in \mathbb{N}}$ into an inverse system of A-modules.

Let P be a finite rank free A-module. We shall often write $P^\vee := \mathrm{Hom}_A(P, A)$. Given any A-module M, there is an isomorphism

$$\mathrm{Hom}_A(P, M) \cong P^\vee \otimes_A M, \tag{3.8}$$

functorial in M and P.

The *dual Koszul complex* associated to the sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ is the complex

$$\mathrm{K}^\vee(A;\boldsymbol{a}) := \mathrm{Hom}_A\big(\mathrm{K}(A;\boldsymbol{a}), A\big). \tag{3.9}$$

This is complex of finite rank free A-modules, concentrated in degrees $0, \ldots, n$. Indeed, for a single element a there is a canonical isomorphism of complexes

$$\mathrm{K}^\vee(A;a) \cong \big(\cdots \to 0 \to A \xrightarrow{a\cdot} A \to 0 \to \cdots \big) \tag{3.10}$$

with A sitting in degrees 0 and 1. And for the sequence we have

$$\mathrm{K}^{\vee}(A;\boldsymbol{a}) \cong \mathrm{K}^{\vee}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{K}^{\vee}(A;a_n).$$

The dual $e_{\boldsymbol{a}}^{\vee} := \mathrm{Hom}(e_{\boldsymbol{a}}, 1_A)$ of $e_{\boldsymbol{a}}$ is a homomorphism of complexes

$$e_{\boldsymbol{a}}^{\vee} : \mathrm{K}^{\vee}(A;\boldsymbol{a}) \to A. \tag{3.11}$$

For any $j \geq i$ in $\mathbb{N}$ there is a homomorphism of complexes

$$p_{\boldsymbol{a},j,i}^{\vee} : \mathrm{K}^{\vee}(A;\boldsymbol{a}^i) \to \mathrm{K}^{\vee}(A;\boldsymbol{a}^j), \tag{3.12}$$

which comes from dualizing the homomorphism (3.6). In this way the collection $\big\{\mathrm{K}^{\vee}(A;\boldsymbol{a}^i)\big\}_{i\in\mathbb{N}}$ becomes a direct system of complexes. The *infinite dual Koszul complex* associated to a sequence $\boldsymbol{a}$ in A is the complex of A-modules

$$\mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a}) := \lim_{i\to} \mathrm{K}^{\vee}(A;\boldsymbol{a}^i). \tag{3.13}$$

For a single element $a \in A$ the infinite dual Koszul complex looks like this: there is a canonical isomorphism

$$\mathrm{K}_{\infty}^{\vee}(A;a) \cong \big(\cdots \to 0 \to A \to A[a^{-1}] \to 0 \to \cdots\big) \tag{3.14}$$

where A is in degree 0, $A[a^{-1}]$ is in degree 1, and the differential $A \to A[a^{-1}]$ is the ring homomorphism. For a sequence we have

$$\mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a}) \cong \mathrm{K}_{\infty}^{\vee}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{K}_{\infty}^{\vee}(A;a_n). \tag{3.15}$$

Thus $\mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a})$ is a complex of flat A-modules concentrated in degrees $0, \ldots, n$.

Let us write

$$e_{\boldsymbol{a},i}^{\vee} : \mathrm{K}^{\vee}(A;\boldsymbol{a}^i) \to A\ ,\ e_{\boldsymbol{a},i}^{\vee} := e_{\boldsymbol{a}^i}^{\vee}\ , \tag{3.16}$$

where $e_{\boldsymbol{a}^i}^{\vee}$ is from (3.11). The homomorphisms $e_{\boldsymbol{a},i}^{\vee}$ respect the direct system, and in the limit we get

$$e_{\boldsymbol{a},\infty}^{\vee} : \mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a}) \to A\ ,\ e_{\boldsymbol{a},\infty}^{\vee} := \lim_{i\to} e_{\boldsymbol{a},i}^{\vee}\ . \tag{3.17}$$

Let $\mathfrak{a}$ be the ideal in A generated by the sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$. From equations (3.14) and (3.15) we see that

$$\mathrm{H}^0\big(\mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a}) \otimes_A M\big) \cong \Gamma_{\mathfrak{a}}(M) \tag{3.18}$$

for any $M \in \mathsf{Mod}\, A$. This gives rise to a functorial homomorphism of complexes

$$v_{\boldsymbol{a},M} : \Gamma_{\mathfrak{a}}(M) \to \mathrm{K}_{\infty}^{\vee}(A;\boldsymbol{a}) \otimes_A M \tag{3.19}$$

that satisfies

$$(e_{\boldsymbol{a},\infty}^{\vee} \otimes 1_M) \circ v_{\boldsymbol{a},M} = \sigma_M \tag{3.20}$$

as homomorphisms $\Gamma_{\mathfrak{a}}(M) \to M$.

An inverse system $\{M_i\}_{i\in\mathbb{N}}$ of abelian groups, with transition maps $p_{j,i} : M_j \to M_i$, is called *pro-zero* if for every i there exists $j \geq i$ such that $p_{j,i}$ is zero. (This is the name used in [Sc].) We shall use the fact that a pro-zero inverse system satisfies the Mittag-Leffler condition. See [We, Definition 3.5.6], where the condition "pro-zero" is called the "trivial Mittag-Leffler" condition.

Definition 3.21. (1) Let $\boldsymbol{a}$ be a finite sequence in a ring A. The sequence $\boldsymbol{a}$ is called *a weakly proregular sequence* if for every $k < 0$ the inverse system $\big\{\mathrm{H}^k(\mathrm{K}(A;\boldsymbol{a}^i))\big\}_{i\in\mathbb{N}}$ (see (3.7)) is pro-zero.

(2) An ideal $\mathfrak{a}$ in a ring A is called a *a weakly proregular ideal* if it is generated by some weakly proregular sequence.

The etymology and history of related concepts are explained in [AJL1] and [Sc]. The next few results are also in found in these papers, but we give the easy proofs for the benefit of the reader.

Example 3.22. A regular sequence $\boldsymbol{a}$ is weakly proregular, since $\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^i)\big) = 0$ for all $i > 0$ and $k < 0$.

Lemma 3.23. *Let $\{M_i\}_{i\in\mathbb{N}}$ be an inverse system of A-modules. The following conditions are equivalent*:

(i) *The system $\{M_i\}_{i\in\mathbb{N}}$ is pro-zero.*
(ii) *For every injective A-module I, $\lim_{i\to} \operatorname{Hom}_A(M_i, I) = 0$.*

Proof. The implication (i) ⇒ (ii) is trivial. For the other direction, take any $i \in \mathbb{N}$, and choose an embedding $\phi : M_i \hookrightarrow I$ for some injective module I. So ϕ is an element of $\operatorname{Hom}_A(M_i, I)$. Since the limit is zero, there is some $j \geq i$ such that $\phi \circ p_{j,i} = 0$. Here $p_{j,i} : M_j \to M_i$ is the transition map. This implies that $p_{j,i} = 0$. □

Theorem 3.24 ([Sc])**.** *Let $\boldsymbol{a}$ be a finite sequence in a ring A. The following conditions are equivalent*:

(i) *The sequence $\boldsymbol{a}$ is weakly proregular.*
(ii) *For any injective module I and any $k > 0$ the A-module $\mathrm{H}^k(\mathrm{K}^\vee_\infty(A;\boldsymbol{a}) \otimes_A I)$ is zero.*

Proof. Take any injective A-module I. We get isomorphisms:

$$\begin{aligned}
\mathrm{H}^k\big(\mathrm{K}^\vee_\infty(A;\boldsymbol{a}) \otimes_A I\big) &\cong^{\Diamond} \mathrm{H}^k\big(\lim_{j\to}\big(\mathrm{K}^\vee(A;\boldsymbol{a}^j) \otimes_A I\big)\big) \\
&\cong^{\Diamond} \lim_{j\to} \mathrm{H}^k\big(\mathrm{K}^\vee(A;\boldsymbol{a}^j) \otimes_A I\big) \cong^{\triangle} \lim_{j\to} \mathrm{H}^k\big(\operatorname{Hom}_A\big(\mathrm{K}(A;\boldsymbol{a}^j), I\big)\big) \\
&\cong^{\heartsuit} \lim_{j\to} \operatorname{Hom}_A\big(\mathrm{H}^{-k}\big(\mathrm{K}(A;\boldsymbol{a}^j)\big), I\big).
\end{aligned}$$

The isomorphisms marked $\Diamond$ are because direct limits commute with tensor products and cohomology; the isomorphism $\triangle$ is by (3.8); and the isomorphism marked $\heartsuit$ is due to Corollary 1.12. By Lemma 3.23 the vanishing of this last limit for every $k > 0$ is equivalent to weak proregularity. □

Corollary 3.25. *Let $\boldsymbol{a}$ be a weakly proregular sequence in A, $\mathfrak{a}$ the ideal generated by $\boldsymbol{a}$, and I a K-injective complex in $\mathsf{C}(\mathsf{Mod}\, A)$. Then the homomorphism*

$$v_{\boldsymbol{a},I} : \Gamma_{\mathfrak{a}}(I) \to \mathrm{K}^\vee_\infty(A;\boldsymbol{a}) \otimes_A I$$

is a quasi-isomorphism.

Proof. By Proposition 1.1(2) we can find a quasi-isomorphism $I \to J$, where J is K-injective and every A-module J^i is injective. Consider the commutative diagram

$$\begin{array}{ccc}
\Gamma_{\mathfrak{a}}(I) & \xrightarrow{v_{\boldsymbol{a},I}} & \mathrm{K}^\vee_\infty(A;\boldsymbol{a}) \otimes_A I \\
\downarrow & & \downarrow \\
\Gamma_{\mathfrak{a}}(J) & \xrightarrow{v_{\boldsymbol{a},J}} & \mathrm{K}^\vee_\infty(A;\boldsymbol{a}) \otimes_A J
\end{array}$$

in $\mathsf{C}(\mathsf{Mod}\,A)$. The vertical arrows are quasi-isomorphisms (for instance because $I \to J$ is a homotopy equivalence). It suffices to prove that $v_{\boldsymbol{a},J}$ is a quasi-isomorphism.

Let us write $F(M) := \Gamma_{\mathfrak{a}}(M)$ and $G(M) := \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A M$ for $M \in \mathsf{Mod}\,A$. We need to show that $v_{\boldsymbol{a},J} : F(J) \to G(J)$ is a quasi-isomorphism. By Proposition 1.9 we may assume that J is a single injective module. In this case we know that $\mathrm{H}^0(v_{\boldsymbol{a},J})$ is bijective; see (3.18). Theorem 3.24 implies that $\mathrm{H}^k(v_{\boldsymbol{a},J})$ is bijective for $k > 0$. And of course

$$\mathrm{H}^k\big(\Gamma_{\mathfrak{a}}(J)\big) = \mathrm{H}^k\big(\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A J\big) = 0$$

for all $k < 0$. Hence $v_{\boldsymbol{a},J}$ is a quasi-isomorphism. □

Corollary 3.26. *Let $\boldsymbol{a}$ be a weakly proregular sequence in A, and $\mathfrak{a}$ the ideal generated by $\boldsymbol{a}$. For any $M \in \mathsf{D}(\mathsf{Mod}\,A)$ there is an isomorphism*

$$v^{\mathrm{R}}_{\boldsymbol{a},M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A M$$

in $\mathsf{D}(\mathsf{Mod}\,A)$. The isomorphism $v^{\mathrm{R}}_{\boldsymbol{a},M}$ is functorial in M, and satisfies

$$(e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_M) \circ v^{\mathrm{R}}_{\boldsymbol{a},M} = \sigma^{\mathrm{R}}_M$$

as morphisms $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$.

Proof. It is enough to consider a K-injective complex $M = I$. We define $v^{\mathrm{R}}_{\boldsymbol{a},I} := v_{\boldsymbol{a},I}$ as in (3.19). Due to equation (3.20) the morphism $v^{\mathrm{R}}_{\boldsymbol{a},I}$ satisfies the parallel derived equation. By Corollary 3.25 the morphism $v^{\mathrm{R}}_{\boldsymbol{a},I}$ is an isomorphism in $\mathsf{D}(\mathsf{Mod}\,A)$. □

The corollary says that the diagram

(3.27)
$$\begin{tikzcd}
\mathrm{R}\Gamma_{\mathfrak{a}}(M) \arrow[r, "v^{\mathrm{R}}_M"] \arrow[dr, "\sigma^{\mathrm{R}}_M"'] & \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A M \arrow[d, "e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_M"] \\
& M
\end{tikzcd}$$

in $\mathsf{D}(\mathsf{Mod}\,A)$ is commutative.

Corollary 3.28. *Let $\mathfrak{a}$ be a weakly proregular ideal in A. Then the functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ has finite cohomological dimension. More precisely, if $\mathfrak{a}$ can be generated by a weakly proregular sequence of length n, then the cohomological dimension of $\mathrm{R}\Gamma_{\mathfrak{a}}$ is at most n.*

Proof. Choose any generating sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ for $\mathfrak{a}$. By Corollary 3.26 there is an isomorphism $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \cong \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A M$ for any $M \in \mathsf{D}(\mathsf{Mod}\,A)$. But the amplitude of the complex $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})$ is n (if A is nonzero). □

Lemma 3.29. *For a finite sequence $\boldsymbol{a}$ of elements of A, the homomorphisms*

$$e^{\vee}_{\boldsymbol{a},\infty} \otimes 1,\ 1 \otimes e^{\vee}_{\boldsymbol{a},\infty} : \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \to \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})$$

are quasi-isomorphisms.

Proof. By symmetry it is enough to look only at

$$1 \otimes e^{\vee}_{\boldsymbol{a},\infty} : \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \to \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}).$$

Write $\boldsymbol{a} = (a_1, \ldots, a_n)$. Since $e^{\vee}_{\boldsymbol{a},\infty} = e^{\vee}_{a_1,\infty} \otimes \cdots \otimes e^{\vee}_{a_n,\infty}$, and since the complexes $\mathrm{K}^{\vee}_{\infty}(A; a_i)$ are K-flat, it is enough to consider the case $n = 1$ and $a = a_1$. Here we have a surjective homomorphism of complexes

$$1 \otimes e^{\vee}_{a,\infty} : \mathrm{K}^{\vee}_{\infty}(A; a) \otimes_A \mathrm{K}^{\vee}_{\infty}(A; a) \to \mathrm{K}^{\vee}_{\infty}(A; a).$$

The kernel is the complex $A[a^{-1}] \xrightarrow{\mathrm{d}} A[a^{-1}]$, concentrated in degrees $1, 2$; and it is acyclic. □

Corollary 3.30. *Let $\mathfrak{a}$ be a weakly proregular ideal in a ring A. For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ the morphism*

$$\sigma^{\mathrm{R}}_{\mathrm{R}\Gamma_{\mathfrak{a}}(M)} : \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(M)) \to \mathrm{R}\Gamma_{\mathfrak{a}}(M)$$

is an isomorphism. Thus the functor

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

is idempotent.

Proof. By Corollary 3.26 we can replace $\sigma^{\mathrm{R}}_{\mathrm{R}\Gamma_{\mathfrak{a}}(M)}$ with

$$e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_{\mathrm{K}^{\vee}_{\infty}} \otimes 1_M : \mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A \mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A M \to \mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A M,$$

where $\boldsymbol{a}$ is any weakly proregular sequence generating $\mathfrak{a}$. Lemma 3.29 says that this is a quasi-isomorphism. □

Corollary 3.31. *The subcategory $\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$ is the essential image of the functor*

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A).$$

Proof. Clear from Corollary 3.30. □

Corollary 3.32. *There is equality*

$$\mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}} = \mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A).$$

In other words, a complex M is cohomologically $\mathfrak{a}$-torsion if and only if all its cohomology modules $\mathrm{H}^i(M)$ are $\mathfrak{a}$-torsion.

Proof. One inclusion is clear – see (2.12). For the other direction, we have to show that if $M \in \mathsf{D}_{\mathfrak{a}\text{-tor}}(\mathsf{Mod}\, A)$ then σ^{R}_M is an isomorphism. By Corollary 3.26 we can replace σ^{R}_M with

$$e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_M : \mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A M \to M,$$

where $\boldsymbol{a}$ is any weakly proregular sequence generating $\mathfrak{a}$. The way-out argument of [RD, Proposition I.7.1] says we can assume M is a single $\mathfrak{a}$-torsion module. But then $\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a})^i \otimes_A M = 0$ for all $i > 0$, so $e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_M$ is an isomorphism of complexes. □

Here is an example showing that a similar statement for cohomologically complete complexes is false.

Example 3.33. Let $A := \mathbb{K}[[t]]$, the power series ring in the variable t over a field $\mathbb{K}$, and $\mathfrak{a} := (t)$. As shown in [Ye2, Example 3.20], there is a complex

$$P = \big(\cdots \to 0 \to P^{-1} \xrightarrow{\mathrm{d}} P^0 \to 0 \to \cdots \big)$$

in which P^{-1} and P^0 are $\mathfrak{a}$-adically free A-modules (both are $\mathfrak{a}$-adic completions of a countable rank free A-module), $\mathrm{H}^{-1}(P) = 0$, and the module $\mathrm{H}^0(P)$ *is not* $\mathfrak{a}$*-adically complete*.

On the other hand, the complex P *is* $\mathfrak{a}$*-adically cohomologically complete*. To see this, we note that the A-modules P^i are flat (see [Ye2, Theorem 3.4]), and hence by Proposition 2.6 the morphism $\xi_P : \mathrm{L}\Lambda_{\mathfrak{a}}(P) \to \Lambda_{\mathfrak{a}}(P)$ is an isomorphism. On the other hand the modules P^i are $\mathfrak{a}$-adically complete, so $\tau_P : P \to \Lambda_{\mathfrak{a}}(P)$ is an isomorphism. Therefore $\tau^{\mathrm{L}}_P : P \to \mathrm{L}\Lambda_{\mathfrak{a}}(P)$ is an isomorphism.

Theorem 3.34 ([Sc])**.** *If A is noetherian, then every finite sequence in A is weakly proregular, and every ideal in A is weakly proregular.*

Proof. It is enough to prove that every finite sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ is weakly proregular. In view of Theorem 3.24, it suffices to prove that for any injective module I and any $k > 0$ the A-module $\mathrm{H}^k(\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A I)$ is zero.

We use the structure theory for injective modules over noetherian rings. Because cohomology and tensor product commute with infinite direct sums, it suffices to consider an indecomposable injective A-module; so assume I is the injective hull of $A/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$. This is a $\mathfrak{p}$-torsion module, and also an $A_{\mathfrak{p}}$-module.

If $\mathfrak{a} \subset \mathfrak{p}$ then each $a_i \in \mathfrak{p}$, so $A[a_i^{-1}] \otimes_A I = 0$. This says that $\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a})^k \otimes_A I = 0$ for all $k > 0$.

Next assume that $\mathfrak{a} \not\subset \mathfrak{p}$. Then for at least one index i we have $a_i \notin \mathfrak{p}$, so that a_i is invertible in $A_{\mathfrak{p}}$. This implies that the homomorphism

$$\mathrm{K}^{\vee}_{\infty}(A; a_i)^0 \otimes_A I \to \mathrm{K}^{\vee}_{\infty}(A; a_i)^1 \otimes_A I$$

is bijective. So the complex $\mathrm{K}^{\vee}_{\infty}(A; a_i) \otimes_A I$ is acyclic. Now

$$\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A I \cong \mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{b}) \otimes_A \mathrm{K}^{\vee}_{\infty}(A; a_i) \otimes_A I,$$

where $\boldsymbol{b}$ is the subsequence of $\boldsymbol{a}$ obtained by deleting a_i. Therefore the complex $\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \otimes_A I$ is acyclic. □

Here is a pretty natural example of a weakly proregular sequence in a non-noetherian ring. There is a follow-up in Example 5.5.

Example 3.35. Let $\mathbb{K}$ be a field, and let A and B be adically complete noetherian $\mathbb{K}$-algebras, with defining ideals $\mathfrak{a}$ and $\mathfrak{b}$ respectively. Take $C := A \otimes_{\mathbb{K}} B$. The ring C is often *not noetherian.*

This happens for instance if $\mathbb{K}$ has characteristic 0, and $A = B := \mathbb{K}[[t]]$, the ring of power series in a variable t. Let $\mathfrak{d} \subset C$ be the kernel of the multiplication map $C = A \otimes_{\mathbb{K}} A \to A$. The ideal $\mathfrak{d}$ is not finitely generated. To see why, note that $\mathfrak{d}/\mathfrak{d}^2 \cong \Omega^1_{A/\mathbb{K}}$, and $L \otimes_C \Omega^1_{A/\mathbb{K}} \cong \Omega^1_{L/\mathbb{K}}$, where $L := \mathbb{K}((t))$. Since $L/\mathbb{K}$ is a separable field extension of infinite transcendence degree, it follows that the rank of $\Omega^1_{L/\mathbb{K}}$ is infinite.

Let's return to the general situation above. Choose finite generating sequences $\boldsymbol{a} = (a_1, \ldots, a_m)$ and $\boldsymbol{b} = (b_1, \ldots, b_n)$ for $\mathfrak{a}$ and $\mathfrak{b}$ respectively. By Theorem 3.34 these sequences are weakly proregular. Consider the sequence

$$\boldsymbol{c} := (a_1 \otimes 1, \ldots, a_m \otimes 1, 1 \otimes b_1, \ldots, 1 \otimes b_n)$$

in C. We claim that this sequence *is weakly proregular.* The reason is that for every i there is a canonical isomorphism of DG algebras

$$\mathrm{K}(A; \boldsymbol{a}^i) \otimes_{\mathbb{K}} \mathrm{K}(B; \boldsymbol{b}^i) \cong \mathrm{K}(C; \boldsymbol{c}^i).$$

By the Künneth formula we get isomorphisms of C-modules

$$\mathrm{H}^k(\mathrm{K}(C;\boldsymbol{c}^i)) \cong \bigoplus_{k\le l\le 0} \mathrm{H}^l(\mathrm{K}(A;\boldsymbol{a}^i)) \otimes_{\mathbb{K}} \mathrm{H}^{k-l}(\mathrm{K}(B;\boldsymbol{b}^i))$$

for every $k \le 0$, compatible with i. Thus for every $k < 0$ the inverse system $\big\{\mathrm{H}^k(\mathrm{K}(C;\boldsymbol{c}^i))\big\}_{i\in\mathbb{N}}$ is pro-zero.

4. The Telescope Complex

The purpose of this section is to prove Theorem 4.21.

Let A be a commutative ring (not necessarily noetherian). For a set X and an A-module M we denote by $\mathrm{F}(X,M)$ the set of all functions $f : X \to M$. This is an A-module in the obvious way. We denote by $\mathrm{F}_{\mathrm{fin}}(X,M)$ the submodule of $\mathrm{F}(X,M)$ consisting of functions with finite support. Note that $\mathrm{F}_{\mathrm{fin}}(X,A)$ is a free A-module with basis the delta functions $\delta_x : X \to A$. (This notation comes from [Ye2].)

Definition 4.1. (1) Given an element $a \in A$, the *telescope complex* $\mathrm{Tel}(A;a)$ is the complex

$$\mathrm{Tel}(A;a) := \big(\cdots \to 0 \to \mathrm{F}_{\mathrm{fin}}(\mathbb{N},A) \xrightarrow{\mathrm{d}} \mathrm{F}_{\mathrm{fin}}(\mathbb{N},A) \to 0 \to \cdots\big)$$

concentrated in degrees 0 and 1. The differential d is

$$\mathrm{d}(\delta_i) := \begin{cases} \delta_0 & \text{if } i = 0, \\ \delta_{i-1} - a\delta_i & \text{if } i \ge 1. \end{cases}$$

(2) Given a sequence $\boldsymbol{a} = (a_1,\ldots,a_n)$ of elements of A, we define

$$\mathrm{Tel}(A;\boldsymbol{a}) := \mathrm{Tel}(A;a_1) \otimes_A \cdots \otimes_A \mathrm{Tel}(A;a_n).$$

Note that $\mathrm{Tel}(A;\boldsymbol{a})$ is a complex of free A-modules, concentrated in degrees $0,\ldots,n$. This complex has an obvious functoriality in $(A;\boldsymbol{a})$.

Recall that for $j \in \mathbb{N}$ we write $[0,j] = \{0,\ldots,j\}$. We view $\mathrm{F}([0,j],A)$ as the free submodule of $\mathrm{F}_{\mathrm{fin}}(\mathbb{N},A)$ with basis $\{\delta_i\}_{i\in[0,j]}$.

Let $j \in \mathbb{N}$. For any $a \in A$ let $\mathrm{Tel}_j(A;a)$ be the subcomplex

$$\mathrm{Tel}_j(A;a) := \big(\cdots \to 0 \to \mathrm{F}([0,j],A) \xrightarrow{\mathrm{d}} \mathrm{F}([0,j],A) \to 0 \to \cdots\big)$$

of $\mathrm{Tel}(A;a)$. For the sequence $\boldsymbol{a} = (a_1,\ldots,a_n)$ we define

$$\mathrm{Tel}_j(A;\boldsymbol{a}) := \mathrm{Tel}_j(A;a_1) \otimes_A \cdots \otimes_A \mathrm{Tel}_j(A;a_n).$$

This is a subcomplex of $\mathrm{Tel}(A;\boldsymbol{a})$. It is clear that

$$\mathrm{Tel}(A;\boldsymbol{a}) = \bigcup_{j\ge 0} \mathrm{Tel}_j(A;\boldsymbol{a}). \tag{4.2}$$

Recall the dual Koszul complex $\mathrm{K}^\vee(A;\boldsymbol{a})$ from formula (3.9). For any $j \ge 0$ we define a homomorphism of complexes

$$w_{a,j} : \mathrm{Tel}_j(A;a) \to \mathrm{K}^\vee(A;a^j) \tag{4.3}$$

as follows, using the presentation (3.10) of $\mathrm{K}^\vee(A;a^j)$. In degree 0 the homomorphism

$$w^0_{a,j} : \mathrm{Tel}_j(A;a)^0 = \mathrm{F}([0,j],A) \to \mathrm{K}^\vee(A;a^j)^0 = A$$

is defined to be

$$w^0_{a,j}(\delta_i) := \begin{cases} 1 & \text{if } i = 0, \\ 0 & \text{if } i \ge 1. \end{cases}$$

In degree 1 the homomorphism

$$w^1_{a,j} : \mathrm{Tel}^j(A;a)^1 = \mathrm{F}([0,j],A) \to \mathrm{K}^\vee(A;a^j)^1 = A$$

is defined to be $w^1_{a,j}(\delta_i) := a^{j-i}$. This makes sense since $i \in [0,j]$.

For a sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ we define

$$w_{\boldsymbol{a},j} := w_{a_1,j} \otimes \cdots \otimes w_{a_n,j} : \mathrm{Tel}_j(A;\boldsymbol{a}) \to \mathrm{K}^\vee(A;\boldsymbol{a}^j). \tag{4.4}$$

The homomorphisms of complexes $w_{\boldsymbol{a},j}$ are functorial in j, so in the direct limit we get a homomorphism of complexes

$$w_{\boldsymbol{a}} := \lim_{j\to} w_{\boldsymbol{a},j} : \mathrm{Tel}(A;\boldsymbol{a}) \to \mathrm{K}^\vee_\infty(A;\boldsymbol{a}). \tag{4.5}$$

Of course $w_{\boldsymbol{a}} = w_{a_1} \otimes \cdots \otimes w_{a_n}$. Let us also define

$$u_{\boldsymbol{a}} : \mathrm{Tel}(A;\boldsymbol{a}) \to A \ , \ u_{\boldsymbol{a}} := e^\vee_{\boldsymbol{a},\infty} \circ w_{\boldsymbol{a}} \ ; \tag{4.6}$$

cf. (3.17).

Lemma 4.7. *The homomorphism $w_{\boldsymbol{a},j}$ is a homotopy equivalence, and the homomorphism $w_{\boldsymbol{a}}$ is a quasi-isomorphism.*

Proof. First consider the case $n = 1$, $A = \mathbb{Z}[t]$, the polynomial ring in the variable t, and $a = t$. The fact that $w_{t,j}$ is a quasi-isomorphism is an easy calculation, once we notice that

$$\mathrm{H}^0\big(\mathrm{Tel}_j(\mathbb{Z}[t];t)\big) = \mathrm{H}^0\big(\mathrm{K}^\vee(\mathbb{Z}[t];t^j)\big) = 0,$$

and

$$\mathrm{H}^1\big(\mathrm{Tel}_j(\mathbb{Z}[t];t)\big) \cong \mathrm{H}^1\big(\mathrm{K}^\vee(\mathbb{Z}[t];t^j)\big) \cong \mathbb{Z}[t]/(t^j).$$

Next, for any $(A;a)$ we have a ring homomorphism $\mathbb{Z}[t] \to A$ sending $t \mapsto a$. Since $w_{a,j}$ is gotten from $w_{t,j}$ by the base change $A \otimes_{\mathbb{Z}[t]} -$, and since $\mathrm{Tel}_j(\mathbb{Z}[t];t)$ and $\mathrm{K}^\vee(\mathbb{Z}[t];t^j)$ are bounded complexes of flat $\mathbb{Z}[t]$-modules, it follows that $w_{a,j}$ is also a quasi-isomorphism.

The flatness argument, with induction, also proves that for sequence $\boldsymbol{a}$ of length $n \geq 2$ the homomorphism $w_{\boldsymbol{a},j}$ is a quasi-isomorphism. Because $\mathrm{Tel}_j(A;\boldsymbol{a})$ and $\mathrm{K}^\vee(A;\boldsymbol{a}^j)$ are bounded complexes of free A-modules, it follows that $w_{\boldsymbol{a},j}$ is a homotopy equivalence.

Finally going to the direct limit preserves exactness, so $w_{\boldsymbol{a}}$ is a quasi-isomorphism. □

Warning: the quasi-isomorphism $w_{\boldsymbol{a}}$ is *not a homotopy equivalence* (except in trivial cases).

Proposition 4.8. *Let $\boldsymbol{a}$ be a weakly proregular sequence in A, and $\mathfrak{a}$ the ideal generated by $\boldsymbol{a}$. For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ there is an isomorphism*

$$v^{\mathrm{R}}_{\boldsymbol{a},M} : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to \mathrm{Tel}(A;\boldsymbol{a}) \otimes_A M$$

in $\mathsf{D}(\mathsf{Mod}\, A)$. The isomorphism $v^{\mathrm{R}}_{\boldsymbol{a},M}$ is functorial in M, and satisfies

$$(u_{\boldsymbol{a}} \otimes 1_M) \circ v^{\mathrm{R}}_{\boldsymbol{a},M} = \sigma^{\mathrm{R}}_M$$

as morphisms $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \to M$.

Proof. Combine Lemma 4.7 and Corollary 3.26. □

Let us denote by $\mathfrak{a}$ the ideal of A generated by the sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$. Recall that $A_j = A/\mathfrak{a}^{j+1}$. Since $\mathfrak{a}^{jn} \subset (\boldsymbol{a}^j) \subset \mathfrak{a}^j$ it follows that the canonical homomorphism

$$\lim_{\leftarrow j} \big(A/(\boldsymbol{a}^{j+1}) \otimes_A M \big) \to \lim_{\leftarrow j} \big(A_j \otimes_A M \big) = \Lambda_{\mathfrak{a}}(M) \tag{4.9}$$

is bijective for any module M.

Let us write

$$\mathrm{Tel}^{\vee}_j(A; \boldsymbol{a}) := \mathrm{Hom}_A\big(\mathrm{Tel}_j(A; \boldsymbol{a}), A\big). \tag{4.10}$$

We refer it as the *dual telescope complex*. Note that $\mathrm{Tel}^{\vee}_j(A; \boldsymbol{a})$ is a complex of finite rank free A-modules, concentrated in degrees $-n, \ldots, 0$. The dual of the homomorphism $w_{\boldsymbol{a},j}$ is

$$w^{\vee}_{\boldsymbol{a},j} : \mathrm{K}(A; \boldsymbol{a}^j) \to \mathrm{Tel}^{\vee}_j(A; \boldsymbol{a}). \tag{4.11}$$

Since $w_{\boldsymbol{a},j}$ is a homotopy equivalence, it follows that $w^{\vee}_{\boldsymbol{a},j}$ is also a homotopy equivalence. Therefore

$$\mathrm{H}^0(w^{\vee}_{\boldsymbol{a},j}) : \mathrm{H}^0(\mathrm{Tel}^{\vee}_j(A; \boldsymbol{a})) \to \mathrm{H}^0(\mathrm{K}(A; \boldsymbol{a}^j))$$

is an isomorphism of A-modules. Define

$$\mathrm{tel}_{\boldsymbol{a},j} : \mathrm{Tel}^{\vee}_j(A; \boldsymbol{a}) \to A/(\boldsymbol{a}^j) \tag{4.12}$$

to be the unique homomorphism of complexes such that

$$\mathrm{H}^0(\mathrm{tel}_{\boldsymbol{a},j}) \circ \mathrm{H}^0(w^{\vee}_{\boldsymbol{a},j})^{-1} : \mathrm{H}^0(\mathrm{K}(A; \boldsymbol{a}^j)) \to A/(\boldsymbol{a}^j)$$

is the canonical A-algebra isomorphism (3.4).

For any $M \in \mathsf{C}(\mathsf{Mod}\, A)$ and $j \in \mathbb{N}$ there is a canonical isomorphism of complexes

$$\mathrm{Hom}_A\big(\mathrm{Tel}_j(A; \boldsymbol{a}), M\big) \cong \mathrm{Tel}^{\vee}_j(A; \boldsymbol{a}) \otimes_A M. \tag{4.13}$$

There is also a canonical isomorphism of complexes

$$\mathrm{Hom}_A\big(\mathrm{Tel}(A; \boldsymbol{a}), M\big) \cong \lim_{\leftarrow j} \mathrm{Hom}_A\big(\mathrm{Tel}_j(A; \boldsymbol{a}), M\big) \tag{4.14}$$

coming from (4.2). We define a homomorphism of complexes

$$\begin{aligned} &\mathrm{tel}_{\boldsymbol{a},M,j} : \mathrm{Hom}_A\big(\mathrm{Tel}_j(A; \boldsymbol{a}), M\big) \to A/(\boldsymbol{a}^j) \otimes_A M\ , \\ &\mathrm{tel}_{\boldsymbol{a},M,j} := \mathrm{tel}_{\boldsymbol{a},j} \otimes 1_M\ , \end{aligned} \tag{4.15}$$

using the isomorphism (4.13).

Definition 4.16. For any $M \in \mathsf{C}(\mathsf{Mod}\, A)$ let

$$\mathrm{tel}_{\boldsymbol{a},M} : \mathrm{Hom}_A\big(\mathrm{Tel}(A; \boldsymbol{a}), M\big) \to \Lambda_{\mathfrak{a}}(M)$$

be the homomorphism of complexes

$$\mathrm{tel}_{\boldsymbol{a},M} := \lim_{\leftarrow j} \mathrm{tel}_{\boldsymbol{a},M,j} = \lim_{\leftarrow j} (\mathrm{tel}_{\boldsymbol{a},j} \otimes 1_M)\ .$$

Here we use the isomorphisms (4.14) and (4.9).

Note that $\mathrm{tel}_{\boldsymbol{a},M}$ is functorial in M.

Remark 4.17. For a module M the homomorphism

$$\mathrm{tel}_{\boldsymbol{a},M} : \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a})^0, M\big) \to \Lambda_{\mathfrak{a}}(M)$$

can be expressed explicitly as an $\mathfrak{a}$-adically convergent power series. First we note that an element $f \in \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a})^0, M\big)$ is the same as a function $f : \mathbb{N}^n \to M$. For $a \in A$ and $i \in \mathbb{N}$ we define the "modified i-th power" $p(a,i) \in A$ to be $p(a,0) := 1$, $p(a,1) := -1$ and $p(a,i) := -a^{i-1}$ if $i \geq 2$. Then

$$\mathrm{tel}_{\boldsymbol{a},M}(f) = \sum_{(i_1,\ldots,i_n)\in\mathbb{N}^n} p(a_1,i_1)\cdots p(a_n,i_n) f(i_1,\ldots,i_n) \in \Lambda_{\mathfrak{a}}(M). \tag{4.18}$$

We shall not require this formula.

Consider the homomorphism of complexes

$$\mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) : M \cong \mathrm{Hom}_A(A,M) \to \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big) \tag{4.19}$$

induced by $u_{\boldsymbol{a}} : \mathrm{Tel}(A;\boldsymbol{a}) \to A$.

Lemma 4.20. *For any $M \in \mathsf{Mod}\, A$ there is equality $\mathrm{tel}_{\boldsymbol{a},M} \circ \mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) = \tau_M$, as homomorphisms $M \to \Lambda_{\mathfrak{a}}(M)$.*

Proof. It suffices to prove that for every $j \geq 0$ there is equality

$$\mathrm{tel}_{\boldsymbol{a},M,j} \circ \mathrm{Hom}(u_{\boldsymbol{a},j}, 1_M) = f_j \circ 1_M$$

as homomorphisms $M \to A/(\boldsymbol{a}^j)$, where $f_j : A \to A/(\boldsymbol{a}^j)$ is the canonical ring homomorphism, and $u_{\boldsymbol{a},j} := e^{\vee}_{\boldsymbol{a},j} \circ w_{\boldsymbol{a},j}$. But everything is functorial in M, so we can restrict attention to $M = A$. Thus we have to show that $\mathrm{tel}_{\boldsymbol{a},j} \circ u^{\vee}_{\boldsymbol{a},j} = f_j$.

Consider the diagram

$$\begin{tikzcd}
A \arrow[r, "e_{\boldsymbol{a},j}"] \arrow[rr, bend left, "f_j"] \arrow[dr, "u^{\vee}_{\boldsymbol{a},j}"'] & \mathrm{K}(A;\boldsymbol{a}^j) \arrow[r, "g_j"] \arrow[d, "w_{\boldsymbol{a},j}"] & A/(\boldsymbol{a}^j) \\
& \mathrm{Tel}^{\vee}_j(A;\boldsymbol{a}) \arrow[ur, "\mathrm{tel}_{\boldsymbol{a},j}"'] &
\end{tikzcd}$$

where g_j is the DG algebra homomorphism. By definition the three triangles are commutative. Hence the whole diagram is commutative. □

Theorem 4.21. *Let A be any ring, let $\boldsymbol{a}$ be a weakly proregular sequence in A, and let P be a flat A-module. Then the homomorphism*

$$\mathrm{tel}_{\boldsymbol{a},P} : \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), P\big) \to \Lambda_{\mathfrak{a}}(P)$$

is a quasi-isomorphism.

Proof. Given an inverse system $\{M_j\}_{j\in\mathbb{N}}$ of complexes of abelian groups, for every integer k there is a canonical homomorphism

$$\psi^k : \mathrm{H}^k\Big(\lim_{\leftarrow j} M_j\Big) \to \lim_{\leftarrow j}\big(\mathrm{H}^k(M_j)\big).$$

By definition of $\mathrm{tel}_{\boldsymbol{a},P}$, for $k=0$ there is a commutative diagram

$$\begin{tikzcd}
\mathrm{H}^k\Big(\mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}),P\big)\Big) \arrow[rr, "\mathrm{H}^k(\mathrm{tel}_{\boldsymbol{a},P})"] \arrow[d, "\cong"'] & & \Lambda_{\mathfrak{a}}(P) \arrow[d, "\cong"] \\
\mathrm{H}^k\Big(\lim_{\leftarrow j}\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)\Big) \arrow[rr, "\mathrm{H}^k(\lim_{\leftarrow j}\ \mathrm{tel}_{\boldsymbol{a},P,j})"] \arrow[d, "\psi^k"'] & & \lim_{\leftarrow j}\big((A/(\boldsymbol{a}^j))\otimes_A P\big) \\
\lim_{\leftarrow j}\ \mathrm{H}^k\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big) \arrow[urr, "\lim_{\leftarrow j}\ \mathrm{H}^k(\mathrm{tel}_{\boldsymbol{a},P,j})"'] & &
\end{tikzcd}$$

The left part of the diagram makes sense for every k. We will prove that:

(1) $\lim_{\leftarrow j}\ \mathrm{H}^k\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)=0$ for all $k\neq 0$.
(2) $\mathrm{H}^0(\mathrm{tel}_{\boldsymbol{a},P,j})$ is bijective for every $j\geq 0$.
(3) ψ^k is bijective for every k.

Together these imply that $\mathrm{H}^k(\mathrm{tel}_{\boldsymbol{a},P})$ is bijective for every k.

There are quasi-isomorphisms

$$w^{\vee}_{\boldsymbol{a},j}:\mathrm{K}(A;\boldsymbol{a}^j)\to\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})$$

that are compatible with j. Since P is flat, according to Corollary 1.12 we get induced isomorphisms

$$\mathrm{H}^k\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)\cong\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^j)\otimes_A P\big)\cong\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)\otimes_A P \tag{4.22}$$

that are compatible with j.

There is a canonical ring isomorphism $\mathrm{H}^0\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)\cong A/(\boldsymbol{a}^j)$. By definition of $\mathrm{tel}_{\boldsymbol{a},j}$, the homomorphism

$$\mathrm{H}^0(\mathrm{tel}_{\boldsymbol{a},j}):\mathrm{H}^0\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\big)\to A/(\boldsymbol{a}^j)$$

is bijective. Hence, using Corollary 1.12 again, we see that $\mathrm{H}^0(\mathrm{tel}_{\boldsymbol{a},P,j})$ is also bijective. This proves (2).

We are given that $\boldsymbol{a}$ is a weakly proregular sequence, which means that the homomorphism

$$\mathrm{H}^k(p_{\boldsymbol{a},j',j}):\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^{j'})\big)\to\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)$$

is zero for $k<0$ and $j'\gg j$. As for $k=0$, we know that

$$\mathrm{H}^0\big(\mathrm{K}(A;\boldsymbol{a}^{j'})\big)\to\mathrm{H}^0\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)$$

is surjective for $j'\geq j$. Of course $\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)=0$ for $k>0$. Thus for every k the inverse system of modules

$$\big\{\mathrm{H}^k\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)\big\}_{j\in\mathbb{N}}\cong\big\{\mathrm{H}^k\big(\mathrm{K}(A;\boldsymbol{a}^j)\big)\otimes_A P\big\}_{j\in\mathbb{N}}$$

satisfies the Mittag-Leffler condition.

The inverse system of complexes $\big\{\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big\}_{j\in\mathbb{N}}$ also satisfies the Mittag-Leffler condition, since it has surjective transition maps. (Warning: see Remark 4.24.) Therefore, by [KS1, Proposition 1.1.24] or [We, Theorem 3.5.8], the homomorphisms

$$\psi^k:\mathrm{H}^k\Big(\lim_{\leftarrow j}\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)\Big)\to\lim_{\leftarrow j}\ \mathrm{H}^k\big(\mathrm{Tel}^{\vee}_j(A;\boldsymbol{a})\otimes_A P\big)$$

are bijective. Thus (3) is true.

Finally, weak proregularity, with the isomorphisms (4.22), tell us that the homomorphism

$$\mathrm{H}^k\big(\mathrm{Tel}^{\vee}_{j'}(A;\boldsymbol{a}) \otimes_A P\big) \to \mathrm{H}^k\big(\mathrm{Tel}^{\vee}_{j}(A;\boldsymbol{a}) \otimes_A P\big)$$

is zero for $k < 0$ and $j' \gg j$. And everything is zero for $k > 0$. This implies (1). $\square$

Corollary 4.23. *Assume $\boldsymbol{a}$ is a weakly proregular sequence in A. Then for every K-flat complex P the homomorphism*

$$\mathrm{tel}_{\boldsymbol{a},P} : \mathrm{Hom}_A(\mathrm{Tel}(A;\boldsymbol{a}), P) \to \Lambda_{\mathfrak{a}}(P)$$

is a quasi-isomorphism.

Proof. By Proposition 1.1 we can assume that P is a complex of flat modules. By Proposition 1.9 we reduce to the case of a single flat module P. This is the theorem above. $\square$

Remark 4.24. The inverse systems of complexes $\big\{\mathrm{K}(A;\boldsymbol{a}^j) \otimes_A P\big\}_{j\in\mathbb{N}}$ does not satisfy the ML condition; so we can't expect to get a quasi-isomorphism in the inverse limit: the homomorphism

$$\lim_{\leftarrow j}\, (w^{\vee}_{\boldsymbol{a},j} \otimes 1_P) : \lim_{\leftarrow j} \big(\mathrm{K}(A;\boldsymbol{a}^j) \otimes_A P\big) \to \lim_{\leftarrow j} \big(\mathrm{Tel}^{\vee}_{j}(A;\boldsymbol{a}) \otimes_A P\big)$$

will usually not be a quasi-isomorphism.

Indeed, this will even fail for the ring $A := \mathbb{K}[t]$, the polynomial algebra over a field $\mathbb{K}$, with sequence $\boldsymbol{a} := (t)$ and flat module $P := A$. Here we get

$$\mathrm{H}^0\big(\lim_{\leftarrow j}\, \mathrm{Tel}^{\vee}_{j}(A;\boldsymbol{a})\big) \cong \mathrm{H}^0\big(\mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), A\big)\big) \cong \Lambda_{\mathfrak{a}}(A) \cong \mathbb{K}[[t]].$$

But $\lim_{\leftarrow j}\, \mathrm{K}(A;\boldsymbol{a}^j)^0 \cong A$ and $\lim_{\leftarrow j}\, \mathrm{K}(A;\boldsymbol{a}^j)^{-1} = 0$, giving

$$\mathrm{H}^0\big(\lim_{\leftarrow j}\, \mathrm{K}(A;\boldsymbol{a}^j)\big) \cong A = \mathbb{K}[t].$$

Corollary 4.25. *Assume $\boldsymbol{a}$ is a weakly proregular sequence in A. For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ there is an isomorphism*

$$\mathrm{tel}^{\mathrm{L}}_{\boldsymbol{a},M} : \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big) \xrightarrow{\simeq} \mathrm{L}\Lambda_{\mathfrak{a}}(M)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$, functorial in M, such that

$$\mathrm{tel}_{\boldsymbol{a},M} \circ \mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) = \tau^{\mathrm{L}}_M,$$

as morphisms $M \to \mathrm{L}\Lambda_{\mathfrak{a}}(M)$.

Proof. It is enough to consider a K-flat complex $M = P$. For this we combine Theorem 4.21, Proposition 2.6 and Lemma 4.20. $\square$

The corollary says that the diagram

(4.26)
$$\begin{tikzcd} M \arrow[d, "\mathrm{Hom}(u_{\boldsymbol{a}},1_M)"'] \arrow[dr, "\tau^{\mathrm{L}}_M"] & \\ \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big) \arrow[r, "\mathrm{tel}^{\mathrm{L}}_{\boldsymbol{a},M}"'] & \mathrm{L}\Lambda_{\mathfrak{a}}(M) \end{tikzcd}$$

is commutative.

Corollary 4.27. *Let $\mathfrak{a}$ be a weakly proregular ideal in A. The cohomological dimension of the functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ is finite. Indeed, if $\mathfrak{a}$ can be generated by a weakly proregular sequence of length n, then the cohomological dimension of $\mathrm{L}\Lambda_{\mathfrak{a}}$ is at most n.*

Proof. This is immediate from Corollary 4.25. □

Remark 4.28. The name "telescope complex" is inspired by a standard construction in algebraic topology; see [GM]. However here we are looking at a specific complex of A-modules, and we prove that it has the expected homological properties.

The result [Sc, Theorem 4.5], which corresponds to our Theorem 4.21, only talks about *bounded complexes* M, and there is an extra assumption that *each* a_i *has bounded torsion.* Moreover, Schenzel states that the question for unbounded complexes is *open as far as he knows.* We answer this in the affirmative in our Theorem 4.21: our result holds for unbounded complexes, and there is no further assumption beyond the weak proregularity of the sequence $\boldsymbol{a}$.

In [AJL1] there is an assertion similar to Theorem 4.21 (more precisely, it corresponds to Theorem 6.12). This is [AJL1, formula $(0.3)_{\mathrm{aff}}$], that also refers to unbounded complexes, and makes no assumption except proregularity of the sequence $\boldsymbol{a}$. In [AJL1, Correction] there is some elaboration on the specific conditions needed for the proofs to be correct. As far as we understand, the correct conditions are weak proregularity for $\boldsymbol{a}$, plus bounded torsion for each a_i. Hence our Theorem 4.21, and also our Theorem 6.12, appear to be stronger than the affine versions of the results in [AJL1].

Our proof of Theorem 4.21 does not depend on any of the results in either [AJL1] or [Sc]. We believe our proof is quite transparent. Note also that we give an explicit formula for the homomorphism of complexes $\mathrm{tel}_{\boldsymbol{a},P}$, that is not found in prior papers.

5. Permanence of Weak Proregularity

In this section we show that weak proregularity is a property of the adic topology defined by an ideal $\mathfrak{a}$; or, otherwise put, it is a property of the closed subset of $\operatorname{Spec} A$ defined by $\mathfrak{a}$.

Theorem 5.1. *Let A be a ring, let $\boldsymbol{a}$ and $\boldsymbol{b}$ be finite sequences of elements of A, and let $\mathfrak{a} := (\boldsymbol{a})$ and $\mathfrak{b} := (\boldsymbol{b})$, the ideals generated by these sequences. Assume that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then $\boldsymbol{a}$ is weakly proregular if and only if $\boldsymbol{b}$ is weakly proregular.*

Proof. For a sufficiently large positive integer p we have $b_i^p \in \mathfrak{a}$ and $a_j^p \in \mathfrak{b}$ for all i, j. Hence there are finitely many $c_{i,j}, d_{j,i} \in A$ such that $b_i^p = \sum_j c_{i,j} a_j$ and $a_j^p = \sum_i d_{j,i} b_i$. Define $\tilde{A}$ to be the quotient of the polynomial ring $\mathbb{Z}[\{s_i, t_j, u_{i,j}, v_{j,i}\}]$ in finitely many variables, modulo the relations $t_i^p = \sum_j u_{i,j} s_j$ and $s_j^p = \sum_i v_{j,i} t_i$. Let $\tilde{a}_i \in \tilde{A}$ and $\tilde{b}_j \in \tilde{A}$ be the images of s_i and t_j respectively. There is a ring homomorphism $f : \tilde{A} \to A$ such that $f(\tilde{a}_i) = a_i$ and $f(\tilde{b_j}) = b_j$.

Define the finite sequences $\tilde{\boldsymbol{a}} := (\tilde{a}_1, \ldots)$ and $\tilde{\boldsymbol{b}} := (\tilde{b}_1, \ldots)$. There are corresponding ideals $\tilde{\mathfrak{a}} := (\tilde{\boldsymbol{a}})$ and $\tilde{\mathfrak{b}} := (\tilde{\boldsymbol{b}})$ in $\tilde{A}$. Since the ring $\tilde{A}$ is noetherian, the sequences $\tilde{\boldsymbol{a}}$ and $\tilde{\boldsymbol{b}}$ are weakly proregular. By construction we have $\sqrt{\tilde{\mathfrak{a}}} = \sqrt{\tilde{\mathfrak{b}}}$, and therefore $\Gamma_{\tilde{\mathfrak{a}}} = \Gamma_{\tilde{\mathfrak{b}}}$ as functors. According to Proposition 4.8 there are isomorphisms

$$\mathrm{Tel}(\tilde{A}; \tilde{\boldsymbol{a}}) \cong \mathrm{R}\Gamma_{\tilde{\mathfrak{a}}}(\tilde{A}) \cong \mathrm{R}\Gamma_{\tilde{\mathfrak{b}}}(\tilde{A}) \cong \mathrm{Tel}(\tilde{A}; \tilde{\boldsymbol{b}})$$

in $\mathsf{D}(\mathsf{Mod}\,\tilde{A})$. Now $\operatorname{Tel}(\tilde{A};\tilde{\boldsymbol{a}})$ and $\operatorname{Tel}(\tilde{A};\tilde{\boldsymbol{b}})$ are bounded complexes of free $\tilde{A}$-modules, so there is a homotopy equivalence $\tilde{\phi} : \operatorname{Tel}(\tilde{A};\tilde{\boldsymbol{a}}) \to \operatorname{Tel}(\tilde{A};\tilde{\boldsymbol{b}})$.

Applying base change along f to $\tilde{\phi}$ we get a homotopy equivalence $\phi : \operatorname{Tel}(A;\boldsymbol{a}) \to \operatorname{Tel}(A;\boldsymbol{b})$ over A. By Lemma 4.7 there are quasi-isomorphisms $w_{\boldsymbol{a}} : \operatorname{Tel}(A;\boldsymbol{a}) \to \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})$ and $w_{\boldsymbol{b}} : \operatorname{Tel}(A;\boldsymbol{b}) \to \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{b})$. Now all these complexes are K-flat; therefore for any A-module I there is a diagram of quasi-isomorphisms

$$\begin{aligned}\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A I &\xleftarrow{w_{\boldsymbol{a}} \otimes 1_I} \operatorname{Tel}(A;\boldsymbol{a}) \otimes_A I \\ &\xrightarrow{\phi \otimes 1_I} \operatorname{Tel}(A;\boldsymbol{b}) \otimes_A I \xrightarrow{w_{\boldsymbol{b}} \otimes 1_I} \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{b}) \otimes_A I.\end{aligned}$$

Taking I to be an arbitrary injective A-module, Theorem 3.24 says that $\boldsymbol{a}$ is weakly proregular if and only if $\boldsymbol{b}$ is weakly proregular. □

Corollary 5.2. *Let $\mathfrak{a}$ be a weakly proregular ideal in a ring A. Then any finite sequence that generates $\mathfrak{a}$ is weakly proregular.*

Proof. Let $\boldsymbol{a}$ be any finite sequence that generates $\mathfrak{a}$. Since $\mathfrak{a}$ is weakly proregular, it has some weakly proregular generating sequence $\boldsymbol{b}$. By the theorem above, $\boldsymbol{a}$ is also weakly proregular. □

Corollary 5.3. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be finitely generated ideals in a ring A, such that $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$. Then $\mathfrak{a}$ is weakly proregular if and only if $\mathfrak{b}$ is weakly proregular.*

Proof. Say $\mathfrak{a}$ is weakly proregular. Choose a weakly proregular generating sequence $\boldsymbol{a}$ for $\mathfrak{a}$. Let $\boldsymbol{b}$ be any finite sequence that generates $\mathfrak{b}$. By the theorem above, $\boldsymbol{b}$ is weakly proregular. Therefore the ideal $\mathfrak{b}$ is weakly proregular. □

Let $f : A \to B$ be a ring homomorphism. There is a forgetful functor (restriction of scalars) $F : \mathsf{Mod}\,B \to \mathsf{Mod}\,A$. Suppose $\mathfrak{a} \subset A$ and $\mathfrak{b} \subset B$ are finitely generated ideals such that $\sqrt{\mathfrak{b}} = \sqrt{B \cdot f(\mathfrak{a})}$ in B. It is easy to see that there are isomorphisms $F \circ \Gamma_{\mathfrak{b}} \cong \Gamma_{\mathfrak{a}} \circ F$ and $F \circ \Lambda_{\mathfrak{b}} \cong \Lambda_{\mathfrak{a}} \circ F$, as functors $\mathsf{Mod}\,B \to \mathsf{Mod}\,A$.

Sometimes such isomorphisms exist also for the derived functors. Note that the forgetful functor F is exact, so it extends to a triangulated functor $F : \mathsf{D}(\mathsf{Mod}\,B) \to \mathsf{D}(\mathsf{Mod}\,A)$.

Theorem 5.4. *Let $f : A \to B$ be a homomorphism of rings, let $\mathfrak{a}$ be an ideal in A, and let $\mathfrak{b}$ be an ideal in B. Assume that the ideals $\mathfrak{a}$ and $\mathfrak{b}$ are weakly proregular, and that $\sqrt{\mathfrak{b}} = \sqrt{B \cdot f(\mathfrak{a})}$. Then there are isomorphisms*

$$F \circ \mathrm{R}\Gamma_{\mathfrak{b}} \cong \mathrm{R}\Gamma_{\mathfrak{a}} \circ F$$

and

$$F \circ \mathrm{L}\Lambda_{\mathfrak{b}} \cong \mathrm{L}\Lambda_{\mathfrak{a}} \circ F$$

of triangulated functors $\mathsf{D}(\mathsf{Mod}\,B) \to \mathsf{D}(\mathsf{Mod}\,A)$.

Proof. In view of Corollary 5.3 we can assume that $\mathfrak{b} = B \cdot f(\mathfrak{a})$. Choose a sequence $\boldsymbol{a} = (a_1, \ldots, a_n)$ that generates $\mathfrak{a}$, and let $\boldsymbol{b} := (f(a_1), \ldots, f(a_n))$. According to Corollary 5.2 the sequences $\boldsymbol{a}$ and $\boldsymbol{b}$ are weakly proregular, in A and B respectively.

We know that $\operatorname{Tel}(B;\boldsymbol{b}) \cong B \otimes_A \operatorname{Tel}(A;\boldsymbol{a})$ as complexes of B-modules. Take any $N \in \mathsf{D}(\mathsf{Mod}\,B)$. Using Corollary 4.25 and Hom-tensor adjunction we get isomorphisms

$$(F \circ \mathrm{L}\Lambda_{\mathfrak{b}})(N) \cong \operatorname{Hom}_B\big(\operatorname{Tel}(B;\boldsymbol{b}), N\big) \cong \operatorname{Hom}_A\big(\operatorname{Tel}(A;\boldsymbol{a}), N\big) \cong (\mathrm{L}\Lambda_{\mathfrak{a}} \circ F)(N).$$

Likewise, using Proposition 4.8, there are isomorphisms

$$(F \circ \mathrm{R}\Gamma_{\mathfrak{b}})(N) \cong \mathrm{Tel}(B;\boldsymbol{b}) \otimes_B N \cong \mathrm{Tel}(A;\boldsymbol{a}) \otimes_A N \cong (\mathrm{R}\Gamma_{\mathfrak{a}} \circ F)(N).$$

□

Example 5.5. This is a continuation of Example 3.35. Let us assume that the ring homomorphisms $\mathbb{K} \to A$ and $\mathbb{K} \to B$ are of formally finite type, in the sense of [Ye1]. (In the terminology of [AJL2] these are pseudo finite type homomorphisms.) Let $\mathfrak{c}$ be the ideal in C generated by the sequence $\boldsymbol{c}$, and define $\widehat{C} := \Lambda_{\mathfrak{c}}(C)$. According to [Ye1, Corollary 1.23] the ring $\widehat{C}$ is noetherian, and the homomorphism $\mathbb{K} \to \widehat{C}$ is of formally finite type. (E.g. if $A = \mathbb{K}[[s]]$ and $B = \mathbb{K}[[t]]$, with defining ideals $\mathfrak{a} := (s)$ and $\mathfrak{b} := (t)$, then $\widehat{C} \cong \mathbb{K}[[s,t]]$.) Let us denote by $\hat{\boldsymbol{c}}$ the image of the sequence $\boldsymbol{c}$ in the ring $\widehat{C}$, and by $\hat{\mathfrak{c}}$ the ideal it generates. By Theorem 3.34 the sequence $\hat{\boldsymbol{c}}$ is weakly proregular. Theorem 5.4 says that there are isomorphisms $\mathrm{R}\Gamma_{\mathfrak{c}} \cong \mathrm{R}\Gamma_{\hat{\mathfrak{c}}}$ and $\mathrm{L}\Lambda_{\mathfrak{c}} \cong \mathrm{L}\Lambda_{\hat{\mathfrak{c}}}$ between the derived functors.

6. MGM Equivalence

In this section A is a commutative ring. We do not assume that A is noetherian or complete. Weak proregularity was defined in Definition 3.21. Recall that any finite sequence in a noetherian ring is weakly proregular, and any ideal in a noetherian ring is weakly proregular (Theorem 3.34).

Lemma 6.1. *Let $\boldsymbol{a}$ be a finite sequence in A, let $\mathfrak{a}$ be the ideal generated by $\boldsymbol{a}$, and let M be an A-module. Then the homomorphism*

$$\Lambda_{\mathfrak{a}}(e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_M) : \Lambda_{\mathfrak{a}}\big(\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A M\big) \to \Lambda_{\mathfrak{a}}(M)$$

(*see* (3.17)) *is an isomorphism of complexes.*

Proof. Since $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})^0 = A$, we have $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})^0 \otimes_A M \cong M$. We will prove that $\Lambda_{\mathfrak{a}}(\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})^i \otimes_A M) = 0$ for $i > 0$. Now $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})^i$ is a direct sum of modules $N_{i,j}$, where $N_{i,j}$ is an $A[a_j^{-1}]$-module. Since

$$(A/\mathfrak{a}^k) \otimes_A N_{i,j} \otimes_A M = 0$$

for any $k \in \mathbb{N}$, in the limit we get $\Lambda_{\mathfrak{a}}(N_{i,j} \otimes_A M) = 0$. □

Lemma 6.2. *Let $\mathfrak{a}$ be a weakly proregular ideal in A. For any complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ the morphism*

$$\mathrm{L}\Lambda_{\mathfrak{a}}(\sigma^{\mathrm{R}}_M) : \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(M)) \to \mathrm{L}\Lambda_{\mathfrak{a}}(M)$$

is an isomorphism.

Proof. Choose a weakly proregular generating sequence $\boldsymbol{a}$ for the ideal $\mathfrak{a}$, and a K-flat resolution $P \to M$ in $\mathsf{C}(\mathsf{Mod}\, A)$. The complex $\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A P$ is also K-flat. By Corollary 3.26 and Proposition 2.6, the morphism $\mathrm{L}\Lambda_{\mathfrak{a}}(\sigma^{\mathrm{R}}_M)$ can be replaced by the homomorphism of complexes

$$\Lambda_{\mathfrak{a}}(e^{\vee}_{\boldsymbol{a},\infty} \otimes 1_P) : \Lambda_{\mathfrak{a}}\big(\mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a}) \otimes_A P\big) \to \Lambda_{\mathfrak{a}}(P). \tag{6.3}$$

But by the previous lemma, the homomorphism (6.3) is actually an isomorphism in $\mathsf{C}(\mathsf{Mod}\, A)$. □

Lemma 6.4. *Let $\boldsymbol{b} = (b_1, \ldots, b_n)$ be a sequence of nilpotent elements in a ring B. Then $u_{\boldsymbol{b}} : \mathrm{Tel}(B;\boldsymbol{b}) \to B$ is a homotopy equivalence.*

Proof. Recall that $u_{\boldsymbol{b}} = e^{\vee}_{\boldsymbol{b},\infty} \circ w_{\boldsymbol{b}}$, where $w_{\boldsymbol{b}} : \mathrm{Tel}(B;\boldsymbol{b}) \to \mathrm{K}^{\vee}_{\infty}(B;\boldsymbol{b})$ is a quasi-isomorphism. By formulas (3.14) and (3.15) we see that $\mathrm{K}^{\vee}_{\infty}(B;\boldsymbol{b})^i = 0$ for $i > 0$, so $e^{\vee}_{\boldsymbol{b},\infty}$ is an isomorphism. We conclude that $u_{\boldsymbol{b}} : \mathrm{Tel}(B;\boldsymbol{b}) \to B$ is a quasi-isomorphism. But these are bounded complexes of free B-modules, and hence $u_{\boldsymbol{b}}$ is a homotopy equivalence. □

Lemma 6.5. *Let $\boldsymbol{a}$ be a finite sequence in A, and let $B := A/(\boldsymbol{a}^j)$ for some $j \geq 1$. Let N be a complex of A-modules, whose cohomology $\mathrm{H}(N)$ is bounded, and such that each $\mathrm{H}^k(N)$ is a B-module. Then the homomorphism*

$$\mathrm{Hom}(u_{\boldsymbol{a}}, 1_N) : N \to \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), N\big)$$

is a quasi-isomorphism.

Proof. Using smart truncation and induction on $\mathrm{amp}(\mathrm{H}(N))$, i.e. by the way-out argument of [RD, Proposition I.7.1], we may assume that N is a single B-module.

Let $\boldsymbol{b}$ denote the image of the sequence $\boldsymbol{a}$ in B. Then $\mathrm{Tel}(B;\boldsymbol{b}) \cong B \otimes_A \mathrm{Tel}(A;\boldsymbol{a})$ as complexes. By Hom-tensor adjunction there is an isomorphism of complexes

$$\mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), N\big) \cong \mathrm{Hom}_B\big(\mathrm{Tel}(B;\boldsymbol{b}), N\big).$$

It suffices then to prove that

$$\mathrm{Hom}(u_{\boldsymbol{b}}, 1_N) : N \cong \mathrm{Hom}_B(B, N) \to \mathrm{Hom}_B\big(\mathrm{Tel}(B;\boldsymbol{b}), N\big)$$

is a quasi-isomorphism. By Lemma 6.4 we know that $u_{\boldsymbol{b}}$ is a homotopy equivalence; and therefore $\mathrm{Hom}(u_{\boldsymbol{b}}, 1_N)$ is a quasi-isomorphism. □

Lemma 6.6. *Let $\mathfrak{a}$ be a weakly proregular ideal in A. For any complex $M \in \mathsf{D}(\mathsf{Mod}\, A)$ the morphism*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(\tau^{\mathrm{L}}_M) : \mathrm{R}\Gamma_{\mathfrak{a}}(M) \to \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M))$$

is an isomorphism.

Proof. By Corollary 4.25 we can replace τ^{L}_M with

$$\mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) : M \to \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big)$$

And by Proposition 4.8 we can replace $\mathrm{R}\Gamma_{\mathfrak{a}}(\tau^{\mathrm{L}}_M)$ with

$$\begin{aligned} 1_{\mathrm{Tel}} \otimes \mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) &: \mathrm{Tel}(A;\boldsymbol{a}) \otimes_A M \\ &\to \mathrm{Tel}(A;\boldsymbol{a}) \otimes_A \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big). \end{aligned} \tag{6.7}$$

We will prove that (6.7) is a quasi-isomorphism.

In view of Proposition 1.9 we can assume that M is a single A-module. Since direct limits commute with cohomology, it suffices to prove that

$$\begin{aligned} 1_{\mathrm{Tel}_j} \otimes \mathrm{Hom}(u_{\boldsymbol{a}}, 1_M) &: \mathrm{Tel}_j(A;\boldsymbol{a}) \otimes_A M \\ &\to \mathrm{Tel}_j(A;\boldsymbol{a}) \otimes_A \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), M\big). \end{aligned} \tag{6.8}$$

is a quasi-isomorphism for every j. Now $\mathrm{Tel}_j(A;\boldsymbol{a})$ is a bounded complex of finite rank free A-modules, so we can replace (6.8) with

$$\mathrm{Hom}(u_{\boldsymbol{a}}, 1_N) : N \to \mathrm{Hom}_A\big(\mathrm{Tel}(A;\boldsymbol{a}), N\big),$$

where $N := \mathrm{Tel}_j(A;\boldsymbol{a}) \otimes_A M$. The complex N satisfies the assumption of Lemma 6.5, and therefore $\mathrm{Hom}(u_{\boldsymbol{a}}, 1_N)$ is a quasi-isomorphism. □

Lemma 6.9. *For a finite sequence $\boldsymbol{a}$ of elements of A, the homomorphisms*

$$u_{\boldsymbol{a}} \otimes 1_{\mathrm{Tel}},\ 1_{\mathrm{Tel}} \otimes u_{\boldsymbol{a}} : \mathrm{Tel}(A; \boldsymbol{a}) \otimes_A \mathrm{Tel}(A; \boldsymbol{a}) \to \mathrm{Tel}(A; \boldsymbol{a})$$

are homotopy equivalences.

Proof. Because of Lemmas 3.29 and 4.7 these are quasi-isomorphisms. But a quasi-isomorphism between K-projective complexes is a homotopy equivalence. □

Proposition 6.10. *Let $\mathfrak{a}$ be a weakly proregular ideal in A. For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ the morphism*

$$\tau^{\mathrm{L}}_{\mathrm{L}\Lambda_{\mathfrak{a}}(M)} : \mathrm{L}\Lambda_{\mathfrak{a}}(M) \to \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M))$$

is an isomorphism. So the functor

$$\mathrm{L}\Lambda_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A) \to \mathsf{D}(\mathsf{Mod}\, A)$$

is idempotent.

Proof. Choose some weakly proregular sequence $\boldsymbol{a}$ that generates $\mathfrak{a}$. According to Corollary 4.25 we can replace $\tau^{\mathrm{L}}_{\mathrm{L}\Lambda_{\mathfrak{a}}(M)}$ with

$$\mathrm{Hom}(1_T, \mathrm{Hom}(u_{\boldsymbol{a}}, 1_M)) : \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A\big(T, \mathrm{Hom}_A(T, M)\big),$$

where $T := \mathrm{Tel}(A; \boldsymbol{a})$. Using Hom-tensor adjunction this can be replaced by

$$\mathrm{Hom}(1_T \otimes u_{\boldsymbol{a}}, 1_M) : \mathrm{Hom}_A(T, M) \to \mathrm{Hom}_A(T \otimes_A T, M).$$

By Lemma 6.9 this is a quasi-isomorphism. □

Theorem 6.11 (MGM Equivalence)**.** *Let A be a ring, and let $\mathfrak{a}$ be a weakly proregular ideal in it.*

(1) *For any $M \in \mathsf{D}(\mathsf{Mod}\, A)$ one has $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}(M) \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$.*

(2) *The functor*

$$\mathrm{R}\Gamma_{\mathfrak{a}} : \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}} \to \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$$

is an equivalence, with quasi-inverse $\mathrm{L}\Lambda_{\mathfrak{a}}$.

Proof. (1) This is immediate from the idempotence of the functors $\mathrm{R}\Gamma_{\mathfrak{a}}$ and $\mathrm{L}\Lambda_{\mathfrak{a}}$; see Corollary 3.30 and Proposition 6.10.

(2) By Lemma 6.6 and Definition 2.11, there are functorial isomorphisms

$$M \cong \mathrm{R}\Gamma_{\mathfrak{a}}(M) \cong \mathrm{R}\Gamma_{\mathfrak{a}}(\mathrm{L}\Lambda_{\mathfrak{a}}(M))$$

for $M \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-tor}}$. By Lemma 6.2 and Definition 2.8 there are functorial isomorphisms

$$N \cong \mathrm{L}\Lambda_{\mathfrak{a}}(N) \cong \mathrm{L}\Lambda_{\mathfrak{a}}(\mathrm{R}\Gamma_{\mathfrak{a}}(N))$$

for $N \in \mathsf{D}(\mathsf{Mod}\, A)_{\mathfrak{a}\text{-com}}$. These isomorphisms set up the desired equivalence. □

Here are a couple of related results.

Theorem 6.12 (GM Duality)**.** *Let A be a ring, and $\mathfrak{a}$ a weakly proregular ideal in A. For any $M, N \in \mathsf{D}(\mathsf{Mod}\, A)$ the morphisms*

$$\mathrm{RHom}_A\big(\mathrm{R}\Gamma_{\mathfrak{a}}(M), \mathrm{R}\Gamma_{\mathfrak{a}}(N)\big) \xrightarrow{\mathrm{RHom}(1,\sigma^{\mathrm{R}}_M)} \mathrm{RHom}_A\big(\mathrm{R}\Gamma_{\mathfrak{a}}(M), N\big)$$
$$\xrightarrow{\mathrm{RHom}(1,\tau^{\mathrm{L}}_N)} \mathrm{RHom}_A\big(\mathrm{R}\Gamma_{\mathfrak{a}}(M), \mathrm{L}\Lambda_{\mathfrak{a}}(N)\big) \xleftarrow{\mathrm{RHom}(\sigma^{\mathrm{R}}_M,1)}$$
$$\mathrm{RHom}_A\big(M, \mathrm{L}\Lambda_{\mathfrak{a}}(N)\big) \xleftarrow{\mathrm{RHom}(\tau^{\mathrm{L}}_M,1)} \mathrm{RHom}_A\big(\mathrm{L}\Lambda_{\mathfrak{a}}(M), \mathrm{L}\Lambda_{\mathfrak{a}}(N)\big)$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ are isomorphisms.

Proof. Choose a weakly proregular sequence $\boldsymbol{a}$ that generates $\mathfrak{a}$, and write $T := \mathrm{Tel}(A; \boldsymbol{a})$ and $u := u_{\boldsymbol{a}}$. Next choose a K-projective resolution $P \to M$ and a K-injective resolution $N \to I$. The complex $T \otimes_A P$ is K-projective, and the complex $\mathrm{Hom}_A(T, I)$ is K-injective.

By Corollary 4.25 and Proposition 4.8 we can replace the diagram above with the diagram

$$\mathrm{Hom}_A\big(T \otimes_A P, T \otimes_A I\big) \xrightarrow{\mathrm{Hom}(1,u\otimes 1)} \mathrm{Hom}_A\big(T \otimes_A P, I\big)$$
$$\xrightarrow{\mathrm{Hom}(1,\mathrm{Hom}(u,1))} \mathrm{Hom}_A\big(T \otimes_A P, \mathrm{Hom}_A(T, I)\big) \xleftarrow{\mathrm{Hom}(u\otimes 1,1)}$$
$$\mathrm{Hom}_A\big(P, \mathrm{Hom}_A(T, I)\big) \xleftarrow{\mathrm{Hom}(\mathrm{Hom}(1,u),1)} \mathrm{Hom}_A\big(\mathrm{Hom}_A(T, P), \mathrm{Hom}_A(T, I)\big)$$

in $\mathsf{C}(\mathsf{Mod}\, A)$. We will prove that all these morphisms are quasi-isomorphisms.

Consider the homomorphism of complexes

$$\mathrm{Hom}(u, 1) : T \otimes_A P \to \mathrm{Hom}_A(T, T \otimes_A P).$$

By Corollary 4.25, Proposition 4.8 and Lemma 6.2 this is a quasi-isomorphism. Therefore, by Hom-tensor adjunction and the fact that I is K-injective, we see that $\mathrm{Hom}(1, u \otimes 1)$ is a quasi-isomorphism.

By Lemma 6.9 and Hom-tensor adjunction it follows that $\mathrm{Hom}(1, \mathrm{Hom}(u, 1))$ and $\mathrm{Hom}(u \otimes 1, 1)$ are quasi-isomorphisms.

Finally consider the homomorphism of complexes

$$1 \otimes \mathrm{Hom}(u, 1) : T \otimes_A P \to T \otimes_A \mathrm{Hom}_A(T, P).$$

By Corollary 4.25, Proposition 4.8 and Lemma 6.6 this is a quasi-isomorphism. Therefore, by Hom-tensor adjunction and the fact that I is K-injective, we see that $\mathrm{Hom}(\mathrm{Hom}(1, u), 1)$ is a quasi-isomorphism. □

Corollary 6.13. *There is a functorial isomorphism*

$$\rho^{\mathrm{LR}}_N : \mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{a}}(A), N) \xrightarrow{\simeq} \mathrm{L}\Lambda_{\mathfrak{a}}(N)$$

for $N \in \mathsf{D}(\mathsf{Mod}\, A)$, such that $\rho^{\mathrm{LR}}_N \circ \mathrm{RHom}(\sigma^{\mathrm{R}}_A, 1_N) = \tau^{\mathrm{L}}_N$ as morphisms $N \to \mathrm{L}\Lambda_{\mathfrak{a}}(N)$.

Proof. Take $M := A$ in Theorem 6.12. □

Remark 6.14. Here is a brief historical survey of the material in this paper. GM Duality for derived categories was introduced in [AJL1]. Precursors, in "classical" homological algebra, were in the papers [Ma1], [Ma2] and [GM].

The construction of the total left derived completion functor $\mathrm{L}\Lambda_{\mathfrak{a}}$ was first done in [AJL1]. Recall that [AJL1] dealt with sheaves on a scheme X, where K-projective resolutions are not available, and certain operations work only for quasi-coherent

$\mathcal{O}_X$-modules. Hence there are some technical difficulties that do not arise when working with rings.

The derived torsion functor goes back to work of Grothendieck in the late 1950's (see [LC] and [RD, Chapter IV]). The use of the infinite dual Koszul complex to prove that the functor $\mathrm{R}\Gamma_{\mathfrak{a}}$ has finite cohomological dimension already appears in [AJL1].

The concept of "telescope" comes from algebraic topology, as a device to form the homotopy colimit in triangulated categories. This is how it was treated in [GM]. Its purpose there was the same as in our proof of Theorem 6.12. We give a concrete treatment of the telescope complex, resulting in our Theorem 4.21.

GM Duality (Theorem 6.12) was already proved in [AJL1]. Perhaps because of the complications inherent to the geometric setup, the proofs in [AJL1] are not quite transparent. Moreover, there was a subtle mistake in [AJL1] involving the concept of proregularity, that was discovered by Schenzel (see [AJL1, Correction] and [Sc]). On the other hand, the results in the later paper [Sc] are not as strong as those in [AJL1], and this is quite confusing. See Remark 4.28 for details. One of our aims in this paper is to clarify the foundations of the theory in the algebraic setting.

MGM Equivalence (Theorem 6.11) is present, in essence, already in [AJL2] and [Sc]; but it is not clear if it can be easily deduced from the existing results in those papers. See a discussion of the various statements and proofs in Remark 4.28.

There is a result similar to Theorem 6.11 in [DG], but the relationship is not clear. In [DG] the authors seem to *define* the derived completion and torsion functors to be $\mathrm{Hom}_A(T, M)$ and $T \otimes_A M$ respectively, where $\boldsymbol{a}$ is a finite sequence and $T := \mathrm{Tel}(A; \boldsymbol{a})$. There is no apparent comparison in [DG] of these functors to the derived functors $\mathrm{L}\Lambda_{\mathfrak{a}}(M)$ and $\mathrm{R}\Gamma_{\mathfrak{a}}(M)$ associated to the ideal $\mathfrak{a}$ generated by $\boldsymbol{a}$ (something like Proposition 4.8 and Corollary 4.25). There is also no assumption that A is noetherian, nor any mention of weak proregularity of $\boldsymbol{a}$. The same reservations pertain also to [DGI].

7. Derived Localization

In this final section we give an alternative characterization of cohomologically complete complexes (Theorem 7.6). This result is inspired by the paper [KS2]. See Remark 7.11 for a comparison.

We make this assumption throughout the section: $\boldsymbol{a} = (a_1, \ldots, a_n)$ is a weakly proregular sequence in the ring A, and $\mathfrak{a}$ is the ideal generated by $\boldsymbol{a}$. We do not assume that A is noetherian or $\mathfrak{a}$-adically complete.

There is an additive functor

$$\Gamma_{0/\mathfrak{a}} : \mathsf{Mod}\, A \to \mathsf{Mod}\, A \ , \ \Gamma_{0/\mathfrak{a}}(M) := M / \Gamma_{\mathfrak{a}}(M) \ .$$

The functor $\Gamma_{0/\mathfrak{a}}$ has a right derived functor $\mathrm{R}\Gamma_{0/\mathfrak{a}}$, constructed using K-injective resolutions.

Lemma 7.1. *For $M \in \mathsf{D}(\mathsf{Mod}\, A)$ there is a distinguished triangle*

$$\mathrm{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\sigma^{\mathrm{R}}_M} M \to \mathrm{R}\Gamma_{0/\mathfrak{a}}(M) \xrightarrow{\triangle} \ ,$$

in $\mathsf{D}(\mathsf{Mod}\, A)$, functorial in M.

Proof. Take any K-injective resolution $M \to I$. Consider the exact sequence

$$0 \to \Gamma_{\mathfrak{a}}(I) \xrightarrow{\sigma_I} I \to \Gamma_{0/\mathfrak{a}}(I) \to 0$$

in $\mathsf{C}(\mathsf{Mod}\, A)$. This gives rise to a distinguished triangle $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\sigma_I} I \to \Gamma_{0/\mathfrak{a}}(I) \xrightarrow{\triangle}$ in $\mathsf{D}(\mathsf{Mod}\, A)$, using the cone construction. But the diagram $\Gamma_{\mathfrak{a}}(I) \xrightarrow{\sigma_I} I$ is isomorphic in $\mathsf{D}(\mathsf{Mod}\, A)$ to the diagram $\mathrm{R}\Gamma_{\mathfrak{a}}(M) \xrightarrow{\sigma^{\mathrm{R}}_M} M$, and $\Gamma_{0/\mathfrak{a}}(I) \cong \mathrm{R}\Gamma_{0/\mathfrak{a}}(M)$. □

Lemma 7.2. *The following conditions are equivalent for* $M \in \mathsf{D}(\mathsf{Mod}\, A)$:

(i) *M is cohomologically $\mathfrak{a}$-adically complete.*

(ii) *M is right perpendicular to* $\mathrm{R}\Gamma_{0/\mathfrak{a}}(A)$; *namely* $\mathrm{RHom}_A\big(\mathrm{R}\Gamma_{0/\mathfrak{a}}(A), M\big) = 0$.

Proof. Start with the distinguished triangle

$$\mathrm{R}\Gamma_{\mathfrak{a}}(A) \xrightarrow{\sigma^{\mathrm{R}}_A} A \to \mathrm{R}\Gamma_{0/\mathfrak{a}}(A) \xrightarrow{\triangle}$$

in $\mathsf{D}(\mathsf{Mod}\, A)$ that we have by Lemma 7.1. Now apply the functor $\mathrm{RHom}_A(-, M)$ to it. This gives a distinguished triangle

$$\mathrm{RHom}_A\big(\mathrm{R}\Gamma_{0/\mathfrak{a}}(A), M\big) \to M \xrightarrow{(\sigma^{\mathrm{R}}_A, 1_M)} \mathrm{RHom}_A\big(\mathrm{R}\Gamma_{\mathfrak{a}}(A), M\big) \xrightarrow{\triangle} .$$

According to Corollary 6.13 we can replace this triangle by the isomorphic distinguished triangle

$$\mathrm{RHom}_A\big(\mathrm{R}\Gamma_{0/\mathfrak{a}}(A), M\big) \to M \xrightarrow{\tau^{\mathrm{L}}_M} \mathrm{L}\Lambda_{\mathfrak{a}}(M) \xrightarrow{\triangle} . \tag{7.3}$$

The equivalence of the two conditions is now clear. □

Let $X := \operatorname{Spec} A$; $Z := \operatorname{Spec} A/\mathfrak{a}$, the closed subset of X defined by the ideal $\mathfrak{a}$; $U := X - Z$, an open set in X; and $U_i := \operatorname{Spec} A[a_i^{-1}]$, the affine open set $\{a_i \neq 0\}$ of X. The collection $\boldsymbol{U} := \{U_i\}_{i=1,\ldots,n}$ is an affine open covering of U.

Let $\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$ be the Čech cosimplicial algebra corresponding to this open covering. So

$$\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)^p = \prod_{1 \leq i_0 \leq \cdots\cdots \leq i_p \leq n} \Gamma(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{O}_X).$$

Note that

$$\Gamma(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{O}_X) \cong A[(a_{i_0} \cdots a_{i_p})^{-1}]$$

as A-algebras.

Any cosimplicial algebra B has the standard normalization $\mathrm{N}(B)$, which is a DG algebra. In degree p the abelian group $\mathrm{N}(B)^p$ is the kernel of all the codegeneracy operators. The multiplication is by the Alexander-Whitney formula (which is usually noncommutative!), and the differential is the alternating sum of the coboundary operators. For full details see any book on simplicial methods; or [HY, Section 1].

Definition 7.4. Let $\mathrm{C}(A; \boldsymbol{a}) := \mathrm{N}(\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X))$, the standard normalization of the cosimplicial algebra $\mathrm{C}(\boldsymbol{U}, \mathcal{O}_X)$. The DG A-algebra $\mathrm{C}(A; \boldsymbol{a})$ is called the *derived localization* of A with respect to the sequence of elements $\boldsymbol{a}$.

Observe that $\mathrm{C}(A; \boldsymbol{a})$ is concentrated in degrees $0, \ldots, n-1$; and each

$$\mathrm{C}(A; \boldsymbol{a})^p \cong \prod_{1 \leq i_0 < \cdots\cdots < i_p \leq n} A[(a_{i_0} \cdots a_{i_p})^{-1}]$$

is a flat A-module. If $n = 1$ then $\mathrm{C}(A; \boldsymbol{a}) = A[a_1^{-1}]$. For $n > 1$ the algebra $\mathrm{C}(A; \boldsymbol{a})$ is noncommutative. We denote by $f_{\boldsymbol{a}} : A \to \mathrm{C}(A; \boldsymbol{a})$ the canonical DG algebra homomorphism.

Lemma 7.5. (1) *There is an isomorphism* $\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a})[1] \cong \operatorname{cone}(f_{\boldsymbol{a}})$ *in* $\mathsf{C}(\mathsf{Mod}\, A)$*. The corresponding distinguished triangle in* $\mathsf{K}(\mathsf{Mod}\, A)$ *is*

$$\mathrm{K}^{\vee}_{\infty}(A; \boldsymbol{a}) \xrightarrow{e^{\vee}_{\boldsymbol{a},\infty}} A \xrightarrow{f_{\boldsymbol{a}}} \mathrm{C}(A; \boldsymbol{a}) \xrightarrow{\triangle} .$$

(2) *The homomorphisms*

$$1_{\mathrm{C}} \otimes f_{\boldsymbol{a}},\ f_{\boldsymbol{a}} \otimes 1_{\mathrm{C}} : \mathrm{C}(A; \boldsymbol{a}) \to \mathrm{C}(A; \boldsymbol{a}) \otimes_A \mathrm{C}(A; \boldsymbol{a})$$

are quasi-isomorphisms.

Proof. (1) This is a direct calculation, quite easy.

(2) Since the complexes in the distinguished triangle in part (1) are all K-flat over A, the assertion follows from Lemma 3.29. □

Theorem 7.6. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be a weakly proregular sequence in the ring* A*, and let* $\mathfrak{a}$ *be the ideal generated by* $\boldsymbol{a}$*. The following conditions are equivalent for* $M \in \mathsf{D}(\mathsf{Mod}\, A)$:

(i) M *is cohomologically* $\mathfrak{a}$*-adically complete.*
(ii) $\mathrm{RHom}_A\big(\mathrm{C}(A; \boldsymbol{a}), M\big) = 0$.

Proof. From Lemma 7.5(1), Lemma 7.1 and Corollary 3.26 (applied to $M := A$) we see that there is an isomorphism $\mathrm{R}\Gamma_{0/\mathfrak{a}}(A) \cong \mathrm{C}(A; \boldsymbol{a})$ in $\mathsf{D}(\mathsf{Mod}\, A)$. Now combine this with Lemma 7.2. □

Let $F : \mathsf{D} \to \mathsf{D}'$ be an additive functor between additive categories. Recall that the *essential image* of F is the full subcategory of D' on the objects $N' \in \mathsf{D}'$ such that $N' \cong F(N)$ for some $N \in \mathsf{D}$. The *kernel* of F is the full subcategory of D on the objects $N \in \mathsf{D}$ such that $F(N) \cong 0$.

Proposition 7.7. *When* $\mathfrak{a}$ *is a weakly proregular ideal in* A*, the kernel of the functor* $\mathrm{L}\Lambda_{\mathfrak{a}}$ *equals the kernel of the functor* $\mathrm{R}\Gamma_{\mathfrak{a}}$.

Proof. This is an immediate consequence of the MGM Equivalence (Theorem 6.11). □

For a DG algebra C we denote by $\mathsf{DGMod}\, C$ the category of left DG C-modules, and by $\tilde{\mathsf{D}}(\mathsf{DGMod}\, C)$ the derived category, gotten by inverting the quasi-isomorphisms in $\mathsf{DGMod}\, C$.

Theorem 7.8. *Let* $\boldsymbol{a} = (a_1, \ldots, a_n)$ *be a weakly proregular sequence in the ring* A*, and let* $\mathfrak{a}$ *be the ideal generated by* $\boldsymbol{a}$*. Consider the triangulated functor*

$$F : \tilde{\mathsf{D}}(\mathsf{DGMod}\, \mathrm{C}(A; \boldsymbol{a})) \to \mathsf{D}(\mathsf{Mod}\, A)$$

induced by the DG algebra homomorphism $A \to \mathrm{C}(A; \boldsymbol{a})$.

(1) *The functor* F *is full and faithful.*
(2) *The essential image of* F *equals the kernel of the functor* $\mathrm{L}\Lambda_{\mathfrak{a}}$.

Proof. (1) Let's write $C := \mathrm{C}(A;\boldsymbol{a})$, $\mathsf{D}(C) := \tilde{\mathsf{D}}(\mathsf{DGMod}\, C)$ and $\mathsf{D}(A) := \mathsf{D}(\mathsf{Mod}\, A)$. Take any $N \in \mathsf{DGMod}\, C$. Lemma 7.5(2) implies that $f_{\boldsymbol{a}} \otimes 1_N : N \to C \otimes_A N$ is a quasi-isomorphism. This shows that the functor $G : \mathsf{D}(A) \to \mathsf{D}(C)$, $G(M) := C \otimes_A M$, is right adjoint to F, and it satisfies $G \circ F \cong \mathbf{1}_{\mathsf{D}(C)}$. Hence F is fully faithful.

(2) Let's write $K := \mathrm{K}^{\vee}_{\infty}(A;\boldsymbol{a})$. Take any $M \in \mathsf{D}(A)$. In view of the idempotence of C (namely Lemma 7.5(2)), Proposition 7.7, Corollary 3.26 and the proof of part (1) above, it is enough to show that $K \otimes_A M \cong 0$ iff $M \cong C \otimes_A M$. Now after applying $- \otimes_A M$ to the distinguished triangle in Lemma 7.5(1) we obtain a distinguished triangle

$$K \otimes_A M \to M \to C \otimes_A M \xrightarrow{\triangle}$$

in $\mathsf{D}(A)$. So the conditions are indeed equivalent. □

Remark 7.9. One can show that $\mathsf{D}(A)_{\mathfrak{a}\text{-tor}}$ is a Bousfield localization of $\mathsf{D}(A)$ in the sense of [Ne, Chapter 9]. Here we use the notation from the proof above. Therefore, using Proposition 7.7 and Theorem 7.8, we see that there is an exact sequence of triangulated categories

$$0 \to \mathsf{D}(C) \xrightarrow{F} \mathsf{D}(A) \xrightarrow{\mathrm{R}\Gamma_{\mathfrak{a}}} \mathsf{D}(A)_{\mathfrak{a}\text{-tor}} \to 0.$$

This was already observed in [AJL1, Remark 0.4] and [DG].

Remark 7.10. The scheme $U = X - Z$ is quasi-affine. We denote by $\mathsf{QCoh}\, \mathcal{O}_U$ the category of quasi-coherent $\mathcal{O}_U$-modules. It can be shown that there is a canonical A-linear equivalence of triangulated categories

$$\mathsf{D}(\mathsf{QCoh}\, \mathcal{O}_U) \approx \tilde{\mathsf{D}}(\mathsf{DGMod}\, \mathrm{C}(A;\boldsymbol{a})).$$

Of course in the principal case ($n = 1$) this is a trivial fact. In terms of derived Morita theory, the equivalence above corresponds to the fact that $\mathcal{O}_U$ is a compact generator of $\mathsf{D}(\mathsf{QCoh}\, \mathcal{O}_U)$.

Remark 7.11. In the paper [KS2] the authors consider the special case where $\mathfrak{a}$ is a principal ideal of A, generated by a regular element (i.e. a non-zero-divisor) a. Here the derived localization $\mathrm{C}(A;a)$ is just the commutative ring $A[a^{-1}]$, and the notation of [KS2] for this algebra is A^{loc}. Theorems 7.6 and 7.8 for this case are closely related to [KS2, Corollaries 1.5.7 and 1.5.9]. The *Cohomological Nakayama Theorem* in [PSY2] is inspired by results in [KS2].

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